

AN
INSTITUTION
OF
FLUXIONS:

Containing the
First Principles,
The *Operations*,
With some of
The Uses and Applications
OF THAT
Admirable Method;

According to the Scheme perfix'd to
his Tract of *Quadratures*, by (its
First Inventor) the Incomparable
Sir ISAAC NEWTON.

By HUMPHRY DITTON.

L O N D O N,
Printed by *W. Botham*; for *James Knapton*, at the
Crown in *St. Paul's Church-Yard*. MDCCVI.

TO THE
Learned and Ingenious
Mr. Benjamin Moreland

MATHEMATICKS,
with their Use and Appli-
cation to Natural Philosophy,
Taught by the Author, at the *Hand*
and Pen in Little-Moorfields: Where
are taught Writing, Arithmetick,
and Merchants Accompts; Also
Youth Boarded, by Ralph Snow.



69d

TO THE
Learned and Ingenious
Mr. *Benjamin Moreland.*

WHAT I here offer you (Sir) is a short Account of that admirable Method of Computation ; beyond which, we are perhaps to expect nothing more to assist us in our Enquiries, either into *Geometry* or *Nature*.

That Sublime Invention, which some have endeavour'd to imitate ; others to obscure ; but all must be fond of and admire.

And as all noble Inventions and Discoveries (besides the Honour they bring to their immediate Authors) fix an indelible Character of Glory upon the Countries that first produce them ; so there's no doubt, but ours upon this account will be Famous to all Posterity.

'Tis a pleasant thing to contemplate the several Gradations and Steps made in the Improvement of our *Methods of Investigation*.

The Method of *Exbansions* (which the old Geometers us'd to admirable good Purpose) after a long Tract of Time, gave place to a

The Dedication.

Superiour one; namely, that of *Indivisibles*. And in much less time, was this Improvement it self, improv'd and heighthned into the noble *Arithmetick of Infinites*. And (Geometry being now in hasty Growth towards its Maturity) in a very short time the Method of *Fluxions* appear'd.

But what a wide Step is it from the Method of Exhaustions hither? How Humble and Low was the Glorious Sucture, when the *First Archimedes* left it in comparison to what 'tis now, the *Second* has set his finishing Hand to it? If there are none that consider these things without Wonder and Delight, there are some that go farther, and are ready to ask the Question, whether the next Step will not be to the *Algebra of pure Intelligences*? However that be; 'tis certain, this Method helps a Man to arrive at Truth, by a most *easy, cheap and compendious* way. And it is for that very reason, by so much more *perfect* than all others, by how much the *fewer* (and those more *facile*) Steps, it takes to obtain its end. All the Works of Nature have a charming *simplicity and conciseness* in them; and there is so much of that Beauty here, that this alone were sufficient to recommend it for the genuine Off-spring of *Eternal Truth and Reason*.

I am at no loss (Sir) for a just and valuable reason of inscribing this small Treatise to your self, when I consider the *Interest* you have in the Subject of it.

Besides,

The Dedication.

Besides, your own warm Love to these Studies, 'tis no Compliment to say, that Mathematicks belongs to your Family, and is an Ornament almost inseparable from it.

Your Uncle Sir Samuel Moreland has long since been Famous in the World for Skill in these Sciences; and after him you have kept the noble Learning amongst you, as if you held it by Inheritance. The Sense therefore (Sir) of doing what I have done, is no more, than presenting you with something you have a kind of a *Right* to. Though besides doing that piece of Justice, I am also very willing to lay hold of the same occasion, to testify to the World, how much I esteem the Character of being,

S I R,

Your Friend

and most Humble Servant,

H. DITTON.

THE PREFACE

TO THE
READER.

THE Design of this Book is to assist the Industrious Student, who either has not, or cannot, manage the larger and nobler Pieces which are already extant on this most useful Subject.

Something of this kind, I am sure, was wanting; and if I have but drawn a good Plan towards it, I shall think I have not miss'd my Aim. A finish'd Piece I don't pretend to, but I wou'd hope what is here done, may serve to ground the Beginners in these Matters, and in some measure answer the End of an Institution, or the first Elements of Fluxions. Neither do I think that to lay the Foundations well is an easier Work, than to build and improve upon these Foundations: For when the first Principles of a Science are well digested, a little Sagacity and Industry will do all the rest.

The Method I have pursued here, is first to explain the Nature and Origine of Fluxions; in which I have endeavour'd to keep as close to the Sense of the first Author and Inventor (the Profoundly Learned

The P R E F A C E.

Learned Sir Isaac Newton) as I could: Whose admirable Introduction is the Ground of all.

Next follow some things concerning the Proportion and Relation of Fluxions to one another; under which Head come the Demonstrations of those Vulgar Theorems, of the Proportions of the Fluxions of the Abscisse, Ordinate and Curve line to each other, as also of the Fluxions of Areas, Solids and Surfaces.

After this I proceed to the Notation of Fluxions; and then to the Operations; which I have all along given the Demonstrations of, in order to the deducing with more certainty and clearness, the Rules necessary for Practise. And to render the Business of Powers and Exponents (which is the principal thing that occurs in the Practise of Fluxions) more easie; I have spent some few Pages in the Explication of it, and endeavoured to deduce it from its first Principles.

The Explication of Second, &c. Fluxions comes next in order; wherein the Reader will see that he continues to work still by the same general Rules, that he did before, and that those Operations are therefore only more difficult than the former, because they are a little more tedious and prolix. To this are subjoyn'd some Rules relating to the same Matter, which I think may be of use.

THE PREFACE.

After this I come to the Fluxions of Logarithms and Exponential Quantities, which I have shewn how to find after the Method of the Celebrated Mr. J. Bernouli, and chosen rather to demonstrate the Lemma in order to it, from the Nature of the Logarithmick Series, than from the Property of the Logarithmick Curve: The former way seeming more general, though the latter be for its Truth unexceptionable. Some Examples of these sort of Fluxions found another way, may be seen in their proper places in the Lexicon Technicum, written by the Learned and Ingenious Mr. John Harris.

And thus we come from the Principles of the Direct Method of Fluxions, to those of the Inverse: Here I endeavour to shew by the Analogy of the Direct Process, how the Fundamental Rules for the Inverse, must be form'd. Neither do I believe that there is any other Method, to discover the first Rules for finding Fluents, but what is founded upon that Analogy. There are also some other Things relating to the Fluents of more compounded Fluxional Expressions, with some additional Observations: All which put together, for all that I know, may amount to as much, as an Introduction need pretend to.

*Those that would enquire farther into the Mysteries of the Inverse Method, will do well to consult
the*

The PREFACE.

the Writings of those Excellent and Ingenious Mathematicians, Dr. Cheyne, and Mr. de Moivre; two Gentlemen, whose Names (notwithstanding their warm, though useful Disputes) I will venture to set by one another, and believe that by that time they have done, the World will be obliged to return them thanks for the Improvements that each occasion'd or contributed, to this part of the Science.

After a Methodical Scheme of the several uses of the Direct and Inverse Method, commonly treated of; I subjoin some few Problems, for the young Geometer's Exercise. My Design was too little, and present Compass too narrow, to take in all the Heads of the above-mentioned Scheme; and not taking in the whole, I would not mangle it by taking but a part, especially since Mr. Hayes has already in his large Work, laid together plenty of the best things of this kind, from the most curious Authors.

Amongst these few Problems, the Reader will find that of the Curve of swiftest Descent, propos'd by the excellent Mr. John Bernouli, and which, as 'tis easily prov'd to be the vulgar Cycloid, from a great many different Properties of that Curve, so I have deduc'd it by a Fluxionary Process from that Property; the comprehending of which, I imagin'd, would give my Reader the least trouble.

Add to this, the Problems of the Isochronal Curve (which is a Semicubical Paraboloid) first pro-

The PREFACE.

propos'd by the Incomparable Mr. Leibnitz, and most neatly solv'd by that accute Geometrician Mr. Fatio; whose Analysis I have here us'd. I have shewn also how some Mechanick Theorems of Galileus and Torricellius relating to the Velocities and Times of the descent of heavy Bodies, may be investigated by Fluxions; and I believe from those few Specimens, twill appear that there's hardly any thing in that Science, but what will, if tried, yield to the same Method.

Lastly, I have given a Specimen of the use of Fluxions in Dioptricks and Catoptricks, in the Resolution of the Problem of the Foci, and that by the help of a Lemma, originally owing to our great Sir Isaac Newton: And which, I doubt not, may yet be farther usefully applied. And this is, in short, an account of what is here done.

But there are yet some things, that I would farther advertise the Reader of.

1. As to the Rectification of the Curve at Pag. 202. the constant Quantity $\frac{2}{3} a$ (to be connected with the other Expression there found) is thus determin'd. The Equation was $x \sqrt{a+x} = c \sqrt{a}$, and the flowing Quantity of the Expression $x \sqrt{a+x}$ was to be found. Suppose $\sqrt{a+x}$ to be the Ordinate of a Curve, whose Abscisse is x , and let $\sqrt{a+x} = y$; then is $y x$ the Fluxion of the Area. And if we suppose another Curve in which the subnormal l shall $= y$ the Ordinate of the first Curve, then also shall $l x = y x$; and if we put z the Ordinate of the

Curve,

THE PREFACE.

Curve, whose Subnormal is 1 or y , then from similar Triangles, it will be $l:z::\dot{x}:\dot{z}$, and so $l\dot{z} = z\dot{x}$, or because of $l=y$, $y\dot{x} = z\dot{z}$, and consequently $\frac{1}{2} z z = \text{Fluent of } y\dot{x}$, that is, the Fluents of $x\sqrt{a+x}$ (or the Area of that Curve) which is $= \frac{2a+2x}{3} \sqrt{a+x}$, as far as by this Operation can be determin'd. But then, since in this Equation $\frac{1}{2} z z = \frac{2a+2x}{3} \sqrt{a+x}$, when the Abscisse x is equal to nothing, the value of $\frac{1}{2} z z$ is $\frac{2}{3} a\sqrt{a}$ (as appears by putting $x=0$) it is plain therefore that the Expression $\frac{2a+2x}{3} \sqrt{a+x}$ exceeds the Area of the Curve (the Fluxion of which is $\dot{x}\sqrt{a+x}$) by the determinate Quantity $\frac{2}{3} a\sqrt{a}$. So that then the Equation of the Flowing Quantities is this $\frac{2a+2x}{3} \sqrt{a+x} - \frac{2}{3} a\sqrt{a} = c\sqrt{a}$, and consequently dividing all by $\sqrt{a}$, we have $c = \frac{\frac{2a+2x}{3} \sqrt{a+x} - \frac{2}{3} a}{\sqrt{a}}$.

The reason of which (and of all Operations of like Nature) may be seen more at large in the Learned Mr. Craig's excellent Book of Quadratures, and from him in Mr. Hayes very well likewise explain'd; the Page I can't so readily direct the Reader to at present, being at the writing of this at a distance from all Books of that kind:
But

The PREFACE.

But 'tis sufficient that the *English* Reader may know where to find it in his own Tongue. And I give him expressly to understand, that after the same Method that the constant Quantity (to be connected with the Expression of the Fluent) is determin'd in this particular Example; they may be found, for the other Examples, from Page 137 to Page 144, or any other whatsoever; neither was there any necessity for the particular determination of those constant Quantities; the Operations there, being design'd as Illustrations of the general Rule of finding Flowing Quantities from Fluxions propos'd, and the caution having been given at large, about compleating the Expressions of Fluents by constant Quantities before at Pag. 129.

2. At Coroll. 2d. Pag. 170. 'Tis said, That at the Point *A*, &c. the Fluxion of the Curve-line is $= 0$; the Reason of which is this, that there the Abscisse *x* is coincident with the Radius *r*, a standing Quantity, and consequently its Fluxion $= 0$: And so the Fluxion of the Curve-line (which is at that Point equal to the Fluxion of the Abscisse) equal to nothing also. If any one thinks that I ought rather to have inferr'd that the Fluxion of the Curve-line was at that Point, only equal to the Fluxion of the Abscisse, and not that 'twas equal to nothing; I can only say, that for the Reason abovementioned, I think 'tis demonstrably otherwise.

3. The Intent of the Process at Coroll. 4. Pag. 170, is, supposing the Abscisse of any one Arch,
to

The P R E F A C E.

to be known, to find the *Abscisse* of another Arch, the Fluxion of which latter Arch shall be to the Fluxion of the former Arch, in a given Ratio ; the *Abscisse* also flowing uniformly. 'Tis true, at the beginning of that Coroll. I mention something of the Arches themselves being in the same given Ratio with their Fluxions : But as I am myself dissatisfied with that Expression, and apprehend some consequence of it that will be unjustifiable, so I would have the Reader understand the scope and design of the Operation to be what I have here already explain'd.

4. At N°. 2. Pag. 21. It may upon some considerations be denied that $\frac{0}{m} = 0$, and said, on the other hand, that the Quote will be infinitely, &c. less than nothing ; and then in that Case, I grant, the consequence there inferr'd will not follow. But not to enter here into the Dark Algorithm of Nothings, and infinitely less than Nothings : If we abate this Head, then Nihilum Geometricum will be never the safer. For 'tis simply impossible, that the Quotient of Real Quantity, divided by Real Quantity, should ever be Nothing in a Mathematical Sense. Such a Quotient may possibly be infinitely small, and so be Nothing in a Physical Sense, or Nothing in a Comparative Sense, and with respect to some other Quantity : But that it should be *Purum Putum Nihil*,

The PREFACE.

Nihil, or strictly and absolutely nothing, is (I think) strictly and absolutely False. And certainly be, that speaking of Nihilum, adds the Epithet Geometricum to it, cannot in reason be imagin'd to mean any thing else, but the Purum Putum. This Term Geometrical, does in the very Nature of the Thing, and according to all the usage of the Word, import the utmost Rigour and Accurateness, whatsoever way 'tis applied. It allows no Latitude nor Limitation, nor Exception, but determines the matter to perfect Exactness. So that up- this account, I cannot see that there is the least room left, for any softning interpretation of the Nihilum to come in. But on the other hand, I think the Noble Author must needs allow the strictest Sense of it, in a consistency with his own Notions; for if we take the Nihilum Geometricum with any Latitude, it will then be Aliquid Quantitatis, or ad-Quantitatem pertinens; and if so, it seems then, that there will be an Aliquid infinitely less than his Infinitesimal, which I don't see how he can well grant.

5. For a farther Instance of the Application of the Doctrine at Prob. 7. concerning the Acceleration in the Description of Curvelineal Figures; we may take that Problem, which is Prop. 11. in a Discourse concerning the Motion of Projects written by Capt. Hally, first printed in the Philos. Trans. then in the Miscellanea Curiosa; to which Book and Figure I now refer, viz. Fig. 1. Pag. 324. The Problem

The P R E F A C E.

Problem is to determine the Force or Velocity of a Project in every Point of the Curve it describes. For Instance, to find the proportion of the Velocity at the Point X in the Curve, to the Velocity impress'd at the Point G. Now the Curve GVX is describ'd (we know) by the Composition of two Motions, one of which is uniform in the Direction of GL, and the other an uniformly accelerated one in the direction LX. And then it presently appears, that drawing the Ordinate XY to the Diameter GY (which Ordinate is therefore parallel to the Tangent GL) that the Velocity of the compound Motion in the Curve at the Point X, is to the Velocity of the compounding equable Motion in the direction of YX the Ordinate (or which is all one, the Fluxion of the Curveline, is to the Fluxion of the Ordinate) as the Tangent to the Ordinate, at the Point X; that is as, $OX : TX$; that is, as $GZ : GL$, for $GL = TX$, and $GZ = OX$, from the Nature of the Parabola. So that the Velocities are as the lengths of the Tangents (to the Parabola in those Points) intercepted between any two Diameters. Q. E. I.

I shall trouble the Reader with no more, than only this one Request, that he would forgive what is amiss, supply what is wanting, and improve upon the Plan which I have here drawn of a compendious Institution of Fluxions.

ERRATA.

ERRATA

Page 42 l. 5, for $x^n no x^{n-1}$ read $x^n + no x^{n-1}$
 ibid. l. 9, for $o^3 x n^{n-2} r. o^3 x^{n-3}$. ibid. l. 12,
 for $\frac{n-3n^3+2n^2}{6} r. \frac{n^3-3n^2+2n}{6}$. p. 60 l. 12,
 for $a^2 + 2ax + x r. a^2 + 2ax + x^2$. p. 74 l. 6,
 for $-\frac{1}{2} r. -\frac{1}{2}$. p. 82 l. 4, for $\dot{x}^2 r. \dot{x}^3$. p. 92 l.
 16 for $x^{m-2} \dot{x} r. x^{m-2} \ddot{x}$. p. 102 l. 5, for
 $y^2 \ddot{x} r. y \ddot{x}$. p. 118 l. 13, for $l: a+x r. l: \frac{a+x}{n}$
 ibid l. 18, for $l a+x r. l: a+x$, and for $l a+x$
 $r. l: \frac{a+x}{n}$ p. 120 for $\frac{-m}{l^m + a+x} r. \frac{-m}{l^{n+1} a+x}$
 p. 125 l. 11 for $l: zy r. l: zj$. p. 137 l. 16,
 for $\frac{m}{m+n} a + \frac{m+n}{x^m} r. \frac{m}{m+n} a + x \frac{m+n}{m}$. p.
 143 l. 13, for $za - aa^2 r. z - a^2$. ibid l. 14, for
 $\frac{z}{2\sqrt{za-aa}} r. \frac{z}{2\sqrt{z-aa}}$. p. 171 l. 10 and 11, in-
 stead of the Ark, read the Fluxion of the Ark.
 p. 176 l. 20, for $\frac{dy^m}{b^n} r. \frac{d^ny^m}{b^m}$ p. 187 l. 15, for T
 r. t. p. 202 l. 5, after Vinculum add this Sen-
 omitted: Which expression, notwithstanding ex-
 ceeds the Truth by the constant Quantity $\frac{2}{3}a$; so
 that the true Fluent is $\frac{2a+2x}{3} \sqrt{a+x} - \frac{2}{3}a$,
 ibid. l. 9, after BM add these Words: Which
 Expression exceeds the Truth by the constant Quanti-
 ty $\frac{2}{3}a$; so that the true Rectification is
 $\frac{2a+2x}{3} \sqrt{a+x} - \frac{2}{3}a$.
 AN

A N
INSTITUTION
O F
FLUXIONS.

S E C T I O N I.

Of the Nature of Fluxions.

A R T I C L E I.

IN order to the forming a Just and Right Conception of the Nature of Fluxions, 'tis necessary to state the Notion, of the *Origin and Generation of Mathematical Quantities.*

Here therefore we are to consider Quantities, not as consisting and made up of *very small Parts*, but as described by a *continued uninterrupted Motion*. We are not to imagine them as the Aggregates or Sums Total, of an infinite number of little constituent Elements, but as

B

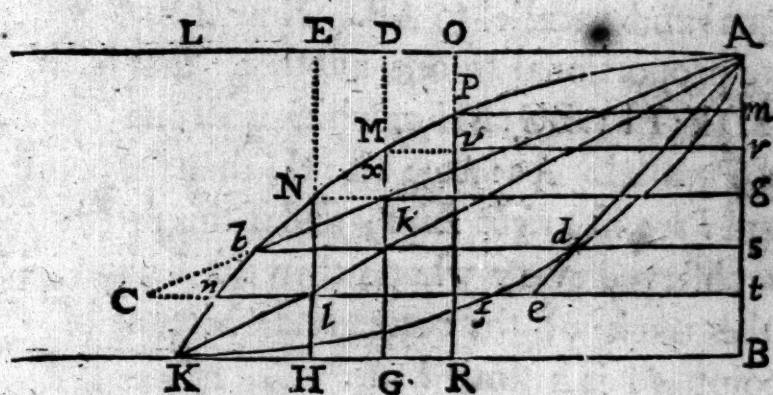
the

the result of a regular Flux, proceeding incessantly, from the first moment of its beginning, to that of perfect rest. A *Line* is describ'd (and in that Description is generated) not by the Apposition of little Lines or Parts, but by the continual motion of a Point. A *Surface* (or plain Figure) is describ'd by the continual Motion of a Line; and a *Solid* by that of a Surface. That Point, or Line, or Surface, which we conceive thus in motion, does by that motion describe (and in describing does generate) a Line, Surface, or Solid: And what sort of Quantity soever it be that we consider, 'tis the best Account of its *Origin* that can be, to say that 'tis describ'd, by this or that Point, Line, or Surface, moving continually in such or such a Direction. Upon this Score it may not be amiss to call a Line, whether strait or crooked, the *Path* mark'd out and describ'd by a Point in motion: A Surface, the *Track* of a moving Line; and tho' the Propriety of Language may hardly allow the use of the same Expression with respect to the Description and Generation of a Solid from the motion of a Surface; yet in effect 'tis the same thing.

A R T. II.

This Matter might easily be illustrated by plenty of Examples ; but considering the plainness of it, one or two may suffice.

A Point may be conceiv'd to be carried with a two-fold sort of Motion ; either a simple one, or a Motion compounded of two others. If we imagine the Points *A* or *B*. (*FIG. 1.*) to



move directly from *A* towards *L*, or from *B* towards *K* ; by that Motion they will describe the right Lines *AL* and *BK*. If the whole Line *AB* moves from *B* towards *K*, and the point *A* at the same instant Commences a Motion from *A* towards *B* ; then the point *A* carried with these two Motions will describe either a right Line, or a Curve, according as the Velocities of it, in each direction are adjusted and proportioned to one another. If they were so proportion'd, as that

B 2

the

the Spaces describ'd (in the same time) by the point *A* in the direction *AB*, and the point *B* in the direction *BK*, were ever directly proportional to one another; then the *Path* of the Motion of the point *A* wou'd be a right Line. For instance, while the point *B* moves from *B* to *G*, suppose *A* to move from *A* to *s*, and when *B* has describ'd the Line *BH*, *A* to have describ'd the Line *At*; that is, when *B* is in *G* or *H*, *A* is in *k* or *l*, in the Lines *DG* and *EH* equal and parallel to *AB*. Now if these Spaces are proportional to one another, that is, if it be $Dk : El :: BG : BH$, or which is all one (because of Parallels) $As : At :: sk : tl$; than the points *k* and *l* are in a right Line; and if this proportion be observed every where, then the Line *AklK* (the track of the compound Motion of the point *A*) is a strait Line.

A R T.

A R T. III.

If the Spaces describ'd in the sametime by the two moving points, be not thus proportional to one another, than the Path of the point A is a Curve Line; either convexe to the Line AB , as the Curve $A d f K$, or concave to the same, as the Curve $A p b K$. In the latter case, the Lines $s b$, and $t n$ are described by the Motion from B to K , in the same time, that the Lines $A s$ and $A t$ are described by the Motion from A to B ; and the Line described by the compound Motion of the point A , is the Curve $A p b K$, concave to the Line AB . In the former case, the Lines $s d$, and $t f$, are described by the Motion from B to K , in the same time that the Lines $A s$, and $A t$ are described by the Motion from A to B ; and the Line describ'd by the compound Motion of the point A , is the Curve $A d f K$, convexe to the Line AB .

A R T. IV.

Now if the Curve $A p b k$ were describ'd by the point A in its compound Motion, than the Spaces describ'd by the Motion in AB would bear a less proportion to each other, than those describ'd by the Motion in BK : And if the Curve $A d f k$ be describ'd, then the Spaces describ'd by the Motion in AB , would bear a greater proportion to each other, than

those describ'd by the Motion in BK. For, let the Subtenfes Ab, and Ad be drawn, intersecting the Line tn in the Points e and c, then from the similar Triangles Asb, Atc,

it shall be $As : At :: Sb : tc$ or $\frac{As}{At} = \frac{sb}{tc}$; but

because $tc > tn$, therefore $\frac{sb}{tn} > \frac{sb}{tc}$; that is,

$\frac{sb}{tn} > \frac{As}{At}$, or the Spaces described by the Motion in the Direction AB, bear a less proportion to each other, than those described in the same times respectively, by the Motion in the Direction BK. On the other hand, with respect to the Curve Adfk, 'tis also $As : At ::$

$ds : te$, and $\frac{As}{At} = \frac{sd}{te}$; but because $te < tf$,

therefore $\frac{sd}{tf} < \frac{sd}{te}$, that is, $\frac{sd}{tf} < \frac{As}{At}$, or the Spa-

ces describ'd by the Motion in the Direction AB, bear a greater Proportion to each other than those describ'd in the same Times respectively in the Direction BK.

A R T. V.

There needs not (after this) any time be spent, in explaining how a Solid is generated, from the continual Motion of a plain Figure; or any sort of Surface from that of a Line.

The

The Intelligent Reader will be able to carry his own Imagination as far as this comes to, without any farther Assistance. I will only add with relation to this business of the Generation and Description of Curve Lines ; that as all the Curves possible to be describ'd or imagin'd, are comprehended within the Limits of this General Division, *viz.* Curves, whose ordinates are all parallel to on another, or else, which all meet in a point, as those ex: gr: of the Spiral Kind : So the Descriptions and Generations of the Curves of both sorts, are to be conceived and accounted for, by the compound Motion of a Point. For, as in the foregoing *Fig* : we supposed the Point *A* to move from *A* towards *B*, while the Line *AB* (retaining still a Position parallel to it self) is carried along the Line *BK* ; which complex Motion of the Point *A* describes a Curve whose Ordinates are all parallel : So if we imagine the Line *AB* to revolve about the fix'd Point *B* as a Center, while the Point *A* moves in the Direction *AB*, from *A* towards *B* (as before) than the Point *A* by these two Motions will describe some one of the *Radial* Curves ; the Ordinates of which (if at least it be proper to call them so) do all meet in the Center *B*. Neither (in both these cases) is there any thing else, but the different Velocities of the two Motions, to be consider'd, as the Fountain of

all that Infinite Variety that we have in Curves.

A R T. VI.

But to proceed; now to the business of Fluxions. The Lines pR , MG , NH , &c. (or BR , BG , BH , &c. any other of the like Nature and Condition) are by a very proper and significant Term called *Flowing Lines*; that is, such as are in continual *Flux* and change, either increasing on the one Hand, or decreasing on the other. While the Line AB is carried from B towards K , by reason of the progress of the Point A from A towards B ; the distance between that Point and the Line BK , is continually varying, and becomes different for every Point of Application between B and K . Thus when AB is come into the Position oR , the Point A must be understood to be in P ; when 'tis in DG , or EH , then the Point A is in M or N ; so that pR , MG , or NH , are the variable distances of the Point A from the Line BK , at those several Moments of Time. From whence the fore-mentioned Lines, pR , MG , NH , &c. may not improperly be called the several *Flowing Values* of the Line AB , for they express the Quantity or Magnitude of it (*in its continual increase or decrease*) when it by its Motion hath proceeded *so far*. And as there is no imaginable

ginable Moment of time wherein the Motions in AB and BK are not going on, so also throughout every imaginable Moment of time, the Line AB is to be conceived as in a *State of Change, and flowing into a new Value*. The same is to be understood of the Lines pM, Mr, Ng, &c. (Ordinates applied to the Axe AB) which are perpetually increasing, as the others were decreasing.

A R T. VII.

Now as to these two Motions (by the Composition of which the Curve Line AMNK is generated) if one of them be supposed to be Uniform and Equable, the other must necessarily be an accelerated one (but accelerated according to such or such Conditions, as the particular Nature of the Curve requires.) If the Motion in AB, ex: gr: be uniform, that in BK must be accelerated; for were both the compounding (rectilineal) Motions equable, the Path describ'd by the Point *A* would be a right Line, as appears by what has been already demonstrated. From whence 'tis certain that the compound Motion it self (with which the describing Point *A* is carried) is not an equable, but some accelerated Motion also. Consequently the *Velocity* is variable for every conceivable Point of the Curve. 'Tis not the
same

same in the next Point or Instant that 'tis in this, but changes without Intermission, and is in continual *Fluxe*. If pM and MN express Particles of the Curve Line (which if generated in equal Particles of time are unequal :) I say, that there are infinite Variations of Velocity to be conceived between the Points p and M , as likewise between the Points M and N , and so of all the rest. Even as there are infinite Variations of Velocity to be conceiv'd between the Points, v , M and x , N ; while those Increments vM and xN (of the Ordinates pm and Mr) are generated by the accelerate Motion from B towards K . And this must needs be so; for as that Motion from B towards K , is (as all Motions in the Generation of Mathematical Quantities must be consider'd to be) continued on without any the least intercepted Stops or Pauses; so being an accelerated one the Acceleration must be without the least imaginable Intermissions too. And for the Acceleration to be without Intermission; is to undergo Infinite and Indeterminate Variations. 'Twould be a contradiction therefore to form the Notion of such a thing as a Velocity, *any where constant, and standing at a stay*; nay, not to conceive it perpetually changing thro' the whole imagin'd Flux of Time.

A R T. VIII.

To assist the Readers Imagination in this matter as much as may be, let us take two small Quantities as A and B , which we will suppose to be the Increments of two several flowing Quantities; the one described by an accelerate, the other by an equable Motion. Let these two Increments A , B , be both describ'd in the same part of time T ; that is, taking a , and b , each very small parts of A and B (and conceiv'd both to be generated in the same Particle of time t) when the time t by flowing is become T , a and b are also become A and B respectively. Likewise expressing any *Velocities* by the Symbols, v , x , y , z , &c: then the Generation of these Increments, the one by the accelerate, the other by the uniform Motion, may be represented to the Eye after this manner.

Flowing Space.	a . $2a$. $3a$. $4a$. &c. A .
Flowing Time.	t . $2t$. $3t$. $4t$. &c. T .
Constant Velocity.	U . U . U . U . &c. U .
Flowing Space.	b . c . d . e . f . &c. B .
Flowing Time.	t . $2t$. $3t$. $4t$. $5t$. &c. T .
Mutable Velocity.	U . v . x . y . z . &c. Z .

Here in the one Case, the little Spaces are proportional to the Times, and the Velocity ever

ver the same, and therefore the former represented by $a. 2a. t. 2t. \&c.$ and the latter by $U.$ which is every where the same Symbol. In the other Case, the little Spaces are not proportionate to the Times $t. 2t. \&c.$ and are therefore express'd by $b. c. d. \&c.$ and the Velocities also ever varying, and therefore represented by a Series of different Symbols, $u, x, y, \&c.$ Tho' in the Cases of both Motions, the result of the Fluxes, is the Generation of the Increments A and B , in the same little part of Time $T.$ What I have said before of the continual variation of Celerities, is, I hope, sufficient to convince the Reader's Judgment, I add this only to help his Fancy.

A R T. IX.

Now to bring this Matter to the main Scope and Purpose : Suppose any flowing Quantities, as the Line, $p m$, $M r$, as also their Increments $M v$ and $N x$ respectively ; which Increments imagine to be generated in equal very small Particles of Time. I conceive we may say without Scruple, that the *Fluxions* are the *Velocities* of those Increments, consider'd not as actually generated, but *quatenus Nascentia*, as arising and beginning to be generated. As there is a vast difference between the Increments consider'd as Finite, or really and actually generated

ed ; and the same considered only as *Nascentia* or in the first Moment of their Generation : So there is as great a difference also between the Velocities of the Increments, consider'd in this two fold respect. And these Distinctions I think ought to be looked upon, as so very Fundamental, and of so much consequence in this Affair ; that without them we can have but very imperfect, if not down-right false and wrong Notions, of the Subject we are upon. That the Notion of a *Fluxion* is to be fixed in *Velocity*, we may learn from the Words of the Noble Inventor of this Method. He tells us (in his Introduction to the Tract of Quadratures) that he sought a *Method of determining Quantities from the Velocities of the Motions or Increments by which they are generated ; and calling those Velocities (of the Motions or Increments) Fluxions, and the Generated Quantities, Fluents, he came at last to the Method which he there uses in the Quadrature of Curves.* And again, in another place (pag. 176) he speaks to this purpose ; *Their Fluxions, or Celerities of encreasing, I express by the same Letters prick'd, &c.* It appears, therefore, to be past all dispute, that the Notion of *Fluxions*, is to be determined to *Velocity*. But then whereas, we say, that the Fluxions are the Velocities of the Increments conceiv'd not as actually Generated, but as *Nascentia* ; as arising, or in a tendency to a Real Exist-

Existence: The Reason of that, is this. Because there is (speaking strictly and accurately) an Infinity of Velocities to be consider'd, in the Generation and Production of a *Real Increment*; as we have shewn in the Discourse before. So that if we conceiv'd the Fluxion, to be the Velocity of the Increment, *as actually Generated*; we must conceive it to be an Infinite Variety or Series of Velocities; which would be to have no clear, distinct, or determinate Notion of it. Whereas the Velocity, with which any sort of Increment *arises*, or *begins to be generated*; is a thing that one may form a very clear and distinct Idea of, and leaves the Mind in no Ambiguity or Confusion at all, as the other does. And to say no more; the *Great Author* tells us, that Fluxions are accurately and exactly in the first Ratio of the Increments consider'd as *Nascentia*; of which see further in the 2d *Corollary* following) and upon this account alone, I think, we may without Hesitation infer, that the Fluxions are therefore the Velocities, not of the Increments consider'd as actually generated, but as *arising*.

A R T. X.

From what has been said concerning the Nature of Fluxions, we may draw the following Conclusions.

1. That

1. That the Genuine Sence and Meaning of finding the Fluxions of any Flowing Quantities, is as much as finding the *Nature* and *Relation of the Velocities* of those Motions, by which the said flowing Quantities are generated and describ'd. And so for the Reverse of this, that from Fluxions propos'd, to find the Flowing Quantities, is the same thing as to proceed from the Relation or Law of the *Generating Velocities* given to determine the Quantities themselves generated and produced thereby.

2. That Fluxions, are accurately and exactly in the first Ratio of the Incrementss, consider'd as arising, or in the first Moment of their Generation. For they are the Velocities with which those *Increments* do so *arise*. Now, as in the Case of Finite Quantities, we demonstrate, That any Spaces describ'd in the same or equal Times, are proportional to the Uniform Velocities by which they are describ'd. So conceiving any Quantities, at the first Moment of their Generation, where all Considerations of Inequality of Time and Acceleration of Motion, are totally and absolutely excluded; they must necessarily *in illo Nascenti statu*, be proportional to the Velocities, with which they do *arise*.

3. Fluxions are *nearly* (and but *nearly*) as the Increments of the flowing Quantities, generated in very small, equal Particles of Time
They

They cannot be accurately as the generated Increments. For they are accurately as the *Nascentia Incrementa*; and the Increments consider'd as *Nascentia*, are not accurately, as the Increments consider'd as *generated* (in determinate Particles of Time.) The *Incrementa Nascentia* are accurately as the Velocities, with which they *arise* and *begin* to be generated; but the Increments *actually* generated in determinate Particles of Time, are not accurately as the Velocities, with which they *arise* and *begin* to be generated. This can't possibly be, unless the Velocities (with which those Increments are generated) were perpetually constant, or every where the same. Then indeed the little Spaces describ'd in a given Particle of Time, would be exactly as those constant Velocities, from the Principles of Mechanicks. But where there is a continued Acceleration or Mutation of Velocity, 'twould be a wrong Conclusion to infer, that the little Spaces describ'd in any equal Particles of Time were exactly as any of those mutable Velocities; or that they were exactly as the Velocities with which they began to be describ'd. We cannot say, therefore, that the Fluxions are *accurately as* the Increments of the flowing Quantities, generated in very small, equal Particles of Time. However, if we take those Particles of time exceeding small indeed, and Neglect the Acceleration

tions of the Velocity as inconsiderable, we may say the Fluxions are proportional to those Increments; remembring at the same time, that they are *but nearly*, and not *accurately* so.

From hence then it appears, that the Notion of Fluxions is mistaken, when we find them represented as the small Increments or Decrements of the Flowing Quantities: 'Tis a mistake upon a two-fold account; for Fluxions in their own Nature, are neither *of* nor *belonging* to Increments, nor are they (accurately speaking) proportional to Increments: And whenever we pretend to represent them by the Increments of the Flowing Quantities, it ought to be with the Limitations and Cautions before-mentioned.

4. The Fundamental Principles which the *Method of Fluxions* is built upon, and proceeds (in all its Operations) from; appear to be more accurate, clear and convincing, than those of the *Differential Calculus*. That these two Methods, (or rather, *this one and the same General Method adorn'd with two several Names*) agree perfectly in all their Operations as to the Point of Practice, is most certain; but it seems to be no less certain and true, that there is a very great difference between them, as to the niceness and accuracy of their first Principles. For this Method, as it goes under the first Name, *viz.* that of the *Method of Fluxions*, proposes to determine Quantities from the Celerities of the

C Motions

Motions by which they are generated ; and those Celerities too, are the Celerities of the Increments, considered as in the first Moment of their Generation. In the other Notion ; the Increments themselves are consider'd purely, without any respect to Velocity, but only imagin'd to be infinitely little, or less than any assignable Quantities. But then ; these Increments, though they are infinitely small, are yet infinitely divisible still. Now as 'twas shewn before, there is an Infinity of Mutable Velocities, to be conceiv'd in the Generation and Production of an Increment, be it as small as it will ; and of all that Series of Velocities 'tis the Velocity, that the Increment *arises* with (or which it has in *statu nascenti*) that the Method of Fluxions considers. And therefore (speaking of exactness in the most strict and rigorous Sence) it appears to be a much higher degree of exactness ; to consider that Celerity of the *Incrementum Nascentis*, beyond which bounds no Imagination can reach ; than to consider the real *Incrementum* it self ; which, tho' Infinitely small, yet makes it necessary to conceive of an infinite Variety of Velocities in the production of it. Again ; if in the *Differential Calculus*, some Terms are rejected and thrown out of an Equation, because they are nothing *Comparatively*, or with respect to other Terms in the same Equation ; that is, because they

they are infinitely small in proportion to those other Terms, and so may be neglected upon that Score : On the other-hand, in the Method of Fluxions, those same Terms go out of the Equation, because they are multiplied into a Quantity, which (upon a most undoubted Principle) does at last really vanish, or become equal to nothing, and so by consequence, all those fore-mentioned Terms into which it is multiplied, do also vanish and become equal to nothing : *Of which Process the Reader will find Examples enough in the latter part of this, and the whole following Section.* Now, I say, the Equation seems to be clear'd of these Terms, after a much cleaner and more satisfactory way, by working according to this latter Process, than by the former. For it seems hardly so consistent with *Geometrical* exactness, to throw Quantities out of an Equation, upon the Score of their being infinitely small or comparatively nothing; as 'tis to reject them because they really vanish, and become equal to nothing *in that Sense.*

N. B. Speaking here of Infinitely small Quantities, or Infinitesimals as some Authors (and particularly Mr. *Neiuentijt*) chuse to term them, I cannot but take notice of a notion, which that Excellent and Ingenious Person advances in his *Analysis Infinitorum*. It is this; That a Quantity Infinitely Great, a Finite or any given Quantity, an Infinitesimal, and *Nihilum Geometricum*,

tricum, are in Geometrick Proportion. I confess I cannot discover the truth of this, though I don't affirm 'tis not true purely upon that account. But I have some Scruples, some reasons of doubting, which I cannot get over, and which for all that I see (at present) are Just.

Let m denote an infinite Quantiy, d any finite one; then is $\frac{d}{m}$ the Infinitesimal of d , according to Mr. *Neiwentit*. Now his Assertion is, that $m : d :: \frac{d}{m} : 0$; therefore since from the nature of Geometrical Proportion, 'tis also $m : d :: \frac{d}{m} : \frac{d d}{m m}$; it follows that $\frac{d d}{m m}$ is $= 0$, or is the *Nihilum Geometricum*. But let it be consider'd.

1. 'Tis plain that $\frac{d d}{m m} = \frac{d}{m} \times \frac{d}{m}$; therefore if $\frac{d}{m} \times \frac{d}{m} = 0$, then $\frac{d}{m} = 0$. Now Mr. *Neiwentit* will hardly allow his Infinitesimal to be nothing; and yet that Infinitesimal $\frac{d}{m}$ is the Side of the Square $\frac{d d}{m m}$; but if the Square itself be equal to nothing, the Side of that Square can't be equal to something; so that if $\frac{d d}{m m} = 0$, $\frac{d}{m} = 0$ also.

2. 'Tis

2. 'Tis plain, that $\frac{0}{m} = 0$, $\frac{0}{m} \times \frac{0}{m} = 0$;
consequently, if $\frac{d}{m} \times \frac{d}{m} = 0$, then $\frac{d}{m} \times \frac{d}{m} =$
 $\frac{0}{m} \times \frac{0}{m}$, from whence I think it must follow,
that $d = 0$.

3. If $\frac{d d}{m m} = 0$, and so $\frac{b b}{m m} = 0$, then $\frac{d d}{m m}$
 $= \frac{b b}{m m}$, and $dd = bb$, and $d = b$. And thus
it seems to me, that by this way of reasoning,
one may prove any Quantity to be equal
to any Quantity; or any finite Quanti-
ty, ever so small, equal to any other finite one,
tho' ever so big.

4. If $\frac{d d}{m m} = 0$, then (dividing by dd) $\frac{1}{m m} =$
 $\frac{0}{dd} = 0$. Therefore $\frac{1}{m m} = \frac{d d}{m m}$, and to $d = 1$,
and so will any other finite Quantity whatso-
ever be prov'd equal to Unity.

I have taken the Liberty to say these few
Things, not with any design to reflect upon a
Person who has deserved so very well of the
Mathematicks, as Mr. Newentiit has; but to
prevent the Mathematical Readers mistake; into
whose Hand, this Author's Excellent Book

may chance to fall some time or other, and where finding a great many noble Truths, he may with less difficulty swallow a Mistake. In my Opinion, Mr. *Neiwentiit* would have concluded more justly, if he had said, that a Quantity infinitely Great, is to a given or finite Quantity, as an Infinitesimal, to an Infinitesimal of another Order or Degree, or to an Infinitesimal infinitely Less than the former Infinitesimal. 'Tis true, this Conclusion wou'd not very well have agreed with another Notion of his, that there are no Differentials of Differentials; that is, in his own Stile, no Infinitesimals of Infinitesimals; or in our Language, no Fluxions of Fluxions. But, however, it would have been more agreeable to the Truth. For Quantity being infinitely Divisible, I think tis most Evident, that not the *Nihilum Geometricum*, but an infinitely smaller Infinitesimal, must be the fourth Proportional to the three above-mentioned.

6. It follows, that in order to find the Relation, between the Fluxions of any Flowing Quantities propos'd, we must take the Increments (of those Flowing Quantities) generated in the same Particle of Time; and then the *first Ratio* of those Increments consider'd as *Nascentia*, or their *ultimate Ratio*, consider'd as *Evanescentia*, will give the Relation of the Fluxions, desired. This is so just and natural
ments

a Consequence from the first Corol. foregoing, that I need say no more of it, upon that Score.

N. B. 'Tis perfectly the same Thing in effect, whether we consider the *Rationes Prima* of the *Incrementa Nascentia*, or the *Rationes Ultima*, or of the *Incrementa Evanescentia*; or (to accommodate the Expression to the more vulgar sort of Readers) whether we take the first Ratio's of the Increments, consider'd as arising, or their last or ultimate Ratio's, when they are considered as vanishing. This, I say, is all one; tho' perhaps sometimes we may proceed to reason from the *Rationes Ultima*, and consider the Increments as *Evanescentia*, and at other times from the *Rationes Prima*, considering the Increments, as *Nascentia*. Which I here take notice of expressly, that the Reader may not be stumbled afterwards.

7. It is manifest that the Progression in Fluxions, is without End or Bounds: That is, that there are Fluxions of Fluxions, and Fluxions of Fluxions, of Fluxions; and so on to Infinity.

The Reason of which is most evident, from what has been hitherto said, of the Generation of Curvilinear Figures. For we see thence that there must needs be a *Continual Acceleration*; or perpetual Mutation of Velocity, all over the Figure, we conceive thus generated and describ'd. So that the *Incrementa* do every where arise or begin to be generated, with va-

riable and different degrees of Velocity. Or in other Words ; the Velocities of the *Incrementa Nascentia*, have their *Mutations*. And for the same reason, that there must be *Mutations* of those Velocities ; there must also necessarily be *Mutations* of the *first Mutations* of Velocity, and consequently, a Series of *Mutations ad Infinitum* ; *Neque enim Novit Natura Limitem*. Whatever, therefore, some may have objected to the contrary, the Progression of Fluxions of *Fluxions*, is demonstratively certain and real in the nature of the thing, and must be admitted by those that admit any Fluxions at all.

SECT.

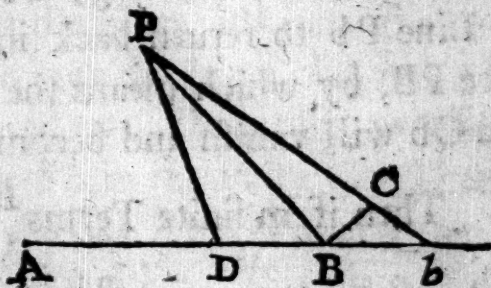
SECTION II.

Of the Relation and Proportion of Fluxions.

FLUXIONS (as we have said) are proportional to, or in the first *Ratio* of the arising Augments, generated in a given Particle of time ; but yet there are right Lines of a *finite length* which they may be shewn to be proportional to, and may consequently be expounded by.

ART. I.

Thus if (FIG. II.) the right Line PB turn-



ing round upon the Point *P* as a Center continually cuts another right Line as *AB*, whose Position is given ; and it be required to find the

the proportion of the Eluxions of these right Lines AB and PB. Suppose the Line PB to move out of the first Place PB, and to come into a new Place Pb. In Pb let us take $PC = PB$, and joining the Points BC, let the Line PD be drawn so as to make the Angle $bPD =$ the Angle bBC ; then will the Triangles bBC , bPD be similar. Now Cb is the Increment of the Line PB, as Bb is that of the Line AB; and both these Increments are evidently generated in the same Particle of Time. For while the Line PB by Flowing is augmented into Pb, the Line AB is also augmented into Ab. From the similar Triangles, it is $Bb : Cb :: Pb : Db$; that is, (since $Pb = PC + Cb$, and $Db = DB + Bb$) $Bb : Cb :: PC + Cb : DB + Bb$; this is the proportion of the finite Augments. Now to obtain the last Ratio of these Augments (consider'd as *vanishing*, or, which is all one, the first Ratio of them consider'd as *arising*) we must imagine the Line Pb to rerurn back into its former place PB, by which means the Augments Bb , and Cb will vanish and become equal to nothing. Then if in finite Terms $\frac{Bb}{Cb}$ be =

$\frac{PC + Cb}{DB + Bb}$ the ultimate Ratio $\frac{Bb}{Cb}$ shall = $\frac{PC}{DB}$

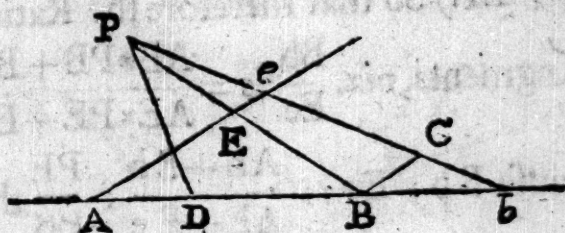
= $\frac{PB}{DB}$, because $PB = PC$. But the Fluxions

are

are in the last Ratio of the evanescent Augments, and consequently the Ratio of the Fluxions of A and PB is $= \frac{PB}{DB}$; that is, they are one to another as PB to DB. Q. E. I.

A R T. II.

Let the right Line PB turning about the



Point *P* as a Center, intersect the two right Lines *AB* and *AE* (given in position) in the Points *E* and *B*. 'Tis required to find the proportion of the Fluxions of those Lines *AB* and *AE*. Suppose as before, the Line *PB* to move from the place *PB* into the new place *Pb*, which cuts the Lines *AB* and *AE*, in the Points *b* and *e*. Draw *BC* parallel to *AE* intersecting *Pb* in the Point *C*. Then are the Triangles *BbC*, *Abe* similar; as also the Triangles *PBC*, *PEe*. 'Tis plain that *Bb* and *Ee*, are the Augments of the Lines *AB* and *AE*, generated in the same time. Now from the similar Triangles *BbC*, *Abe*, 'tis *Bb*: *BC*:
Ab:

$Ab : Ae$; whence $Bb = \frac{BC \times Ab}{Ae}$, and from the

similar Triangles PBC, PEe , 'tis $PB : PE :: BC :$

Ee ; whence $Ee = \frac{PE \times BC}{PB}$. Consequently

$$Bb : Ee :: \frac{BC \times Ab}{Ae} : \frac{BC \times PE}{PB} :: \frac{Ab}{Ae} : \frac{PE}{PB} ::$$

$$\frac{AB + Bb}{AE + Ee} : \frac{PE}{PB} \text{ (for } Ab = AB + Bb, \text{ and } Ae = AE + Ee.) \text{ So that therefore the Ratio of the}$$

finite Augments, viz, $\frac{Bb}{Ee}$ is $= \frac{AB \times PB + Bb \times PB}{AE \times PE + Ee \times PE}$,

for because $Bb : Ee :: \frac{AB + Bb}{AE + Ee} : \frac{PE}{PB}$ therefore

$$Bb : Ee = \frac{AB + Bb}{AE + Ee} : \frac{PE}{PB}, \text{ which is}$$

which is equal to the forementioned Expression.

Therefore the last Ratio $\frac{Bb}{Ee}$ is $= \frac{AB \times PB}{AE \times PE}$

for now the Augments Bb and Ee are suppos'd to vanish, and consequently the Terms drawn into them, vanish also. Therefore the proportion of the Fluxions, is the same with that of these Rectangles; or the Fluxion of AB , to the Fluxion of AE , as $AB \times PB : AE \times PE$. Q; E. I.

Corol. Joining the result of the Conclusions found in this and the former Example; it will appear, that drawing PD so as that the Angle

bBC

to the new place ba . Produce the Lines BO and ba till they meet in N , and draw OM parallel to AB . 'Tis plain that Bb and Oa are the Increments of the Lines AB and AO , generated in the same Particle of time. The Triangles Aab , OaM , are similar; as also the Triangles NBb , NOM . Therefore $OM : Oa :: Ab : Aa$, and $NB : NO :: Bb : OM$. Therefore $Oa : Bb :: \frac{OM \times Aa}{Ab} : \frac{NB \times OM}{NO} :: \frac{Aa}{Ab} : \frac{NB}{NO}$.

$\frac{AO + Oa}{AB + Bb} : \frac{NB}{NO}$. So that the Ratio of the finite

Augments, viz. $\frac{Oa}{Bb}$ is $= \frac{NO \times AO + NO \times Oa}{NB \times AB + NB \times Bb}$.

Therefore ultimate Ratio $\frac{Oa}{Bb} = \frac{NO \times AO}{NB \times AB}$; or,

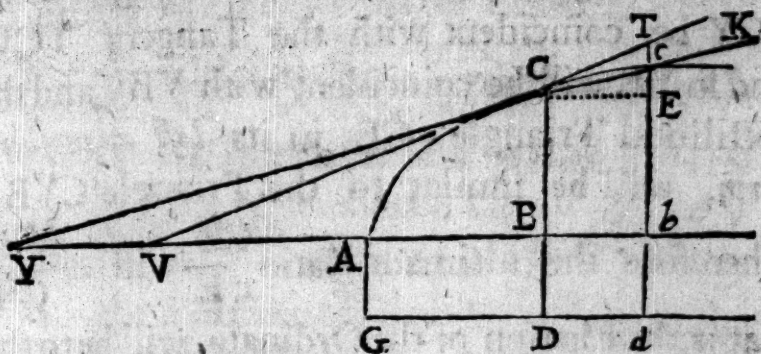
which is all one, the Fluxion of AO is to the Fluxion of AB , as the Rectangle $NO \times AO$, to the Rectangle $NB \times AB$. Q: E. I. Several other Instances of this nature might be proposed, which the ingenious Reader may find out himself, for his own Exercise and Pleasure.

But 'twill be requisite to give some Examples of this matter in Curvilinear Figures, and shew how all those Fundamental Theorems relating to the proportion and Expression of Fluxions, are to be derived and demonstrated from this first Principle we have been applying already.

A R T.

ART. IV.

Let ACc (FIG. V.) be any Curve, whose



Abſciſſe is AB , ordinate CB at Right Angles, and TCV a Tangent at C . 'Tis required to find the relation of the Fluxion of the ordinate to the Fluxion of the Abſciſſe. Let the ordinate BC (ſuppoſed to move uniformly) come into the new place bc ; and drawing CE parallel to AB ; 'tis plain, that the little Lines CE and cE , are the Increments of the Abſciſſe and Ordinate, generated in the ſame Particle of time: For while AB flows into Ab , CB flows into cb . Draw the right Line cC ſubtending the Curvilinear Arch Cc , which Line cC produce till it cuts AB (produced) in the Point Y . The rectilinear Triangles cCE , cYb are ſimilar; therefore $CE : cE :: Yb : cb$, that is, $CE : cE :: YB + Bb$: (or $YB + CE$, for $CE = Bb$) $CB + cE$; therefore in finite Terms, the Ratio of the Augments, viz.

$$\frac{CE}{cE} = \frac{YB + CE}{CB + cE}.$$

Suppoſe now the ordinate

cb

cb to return back into its first place CB, or (which is all one) imagine the Points *c* and *C* to come together; then will the right Line *cCY* be coincident with the Tangent *TCV*, and so *YB* will be coincident with *VB* (and the rectilineal Triangle *cCE* in its *last evanescent form*, will be similar to the Triangle *CVB*.)

Therefore the ultimate Ratio $\frac{CE}{cE}$ will $= \frac{VB}{CB}$; that is, the Fluxion of the Ordinate, will be to the Fluxion of the Abscisse, as the Ordinate *CB*, to the subtangent *VB*. Q: E: I.

2. Let it be propos'd to find the proportion of the Fluxion of the Curve Line *AC*, to the Fluxion of the Ordinate *BC*. The Augments of these flowing Quantities generated in the same time, are the little curvilineal Arch *Cc*, and the little right Line *Ec*. Now arguing with the right Line *Cc* (the Subtense of that little Portion of the Curve *Cc*) from the similarity of the fore-mentioned Triangles, we have *Cc*: *cE*:: *cY*: *cb*, or *Cc*: *cE*:: *cC + CY*: *cE + CB*; so that in finite Terms, $\frac{Cc}{cE} =$

$\frac{cC + CY}{cE + CB}$. But when the Points *c* and *C*

come together, then the Secant *CY* will coincide with the Tangent *CV*; and the evanescent Triangle *CEc*, will in its last form be similar to the Triangle *CVB*; and the sides of the

one be proportional to the sides of the other.

Therefore ultimate Ratio $\frac{\text{Subtense } Cc}{\text{Right Line } cE}$ will =

$\frac{CV}{CB}$. But the ultimate Ratio of the Subtense Cc

to the Curve Cc is a Ratio of Equality ; there-

fore ultimate Ratio $\frac{\text{Curve } Cc}{\text{right Line } cE}$ will = $\frac{CV}{CB}$;

that is, the Fluxion of the Curve, is to the Fluxion of the Ordinate, as the Tangent CV , to the ordinate CB . $Q: E: I$. By the very same way of reasoning it will be found too, that the Fluxion of the Curve Line, is to the Fluxion of the Abscisse, as the Tangent to the Subtangent.

3. Let it be propos'd to find the Proportion of the Fluxion of the Curvilinear *Area* ABC , to the Fluxion of the *Rectangle* $ABGD$. The Augments here generated in the same Particle of time, are the little *Curvilinear Trapezium* $BCcb$, and the little *Parallelogram* $BDbd$; and for Brevities sake, we'll call these Augments A and a , respectively.

Let us express the Curvilinear Space, included between the Curve Cc , and the Line CE , by the Symbol q . Then is the Curvilinear *Trapezium* $BCcb = BC \times Bb + q$; that is, equal to the *Rectangle* $BCEb$ + the Curve Space included between the Arch Cc and the Right Line CE . So that $A: a :: BC \times Bb + q: BD \times Bb$, in

D

finite

finite Terms ; that is, as $BC + \frac{q}{Bb}$: BD . But when the Points C and c come together, then the Space q vanishes. Therefore, ultimate Ratio $\frac{A}{a} = \frac{BC}{BD}$, for $\frac{q}{Bb}$ goes out, and vanishes entirely, and consequently the Fluxion of the Area ABC , is to the Fluxion of the Area $ABGD$, as the ordinate BC to the ordinate BD . Q : E : I .

4. Let it be required to find the proportion of the Fluxion of the *Solid* generated by the Rotation of the Curvilinear Area ABC about the Axe AB , to the Fluxion of the Solid generated by the Rotation of the Rectangle $ABGD$ about the same Axe.

The Augments of these flowing Quantities generated in the same Particle of Time, are the Solids generated by the Rotation of the little Curvilinear Area $BCcb$, and the little Rectangle $BDbd$. Now the Solid generated by the Area $BCcb$ is to be conceived consisting of two others, even as the generating plain Figure consists of two parts, the Rectangle $BCEb$ and the Curvilinear Triangle CcE . So the whole Solid produced by the Rotation consists of the Solid generated by the Rectangle $BCEb$ which is a Cylinder whole Base is BC , the Altitude CE or Bb ; and the Solid generated by the Curvilinear Triangle CEc , which is a sort of a
Ring

Ring or *Annulus*. Using the Symbols A , a , as before ; let q now denote the little solid generated by the Rotation of the Curvilinear Space CcE about the Axe AB . Then shall A and a be expressed by $BCq \times Bb + q$, and $BDq \times Bb$; for the Circles describ'd by BC and BD are as the Squares of those Lines respectively, and so $BCq \times Bb$ is to the Cylinder describ'd by the Rectangle $BCEb$, as $BDq \times Bb$, to the Cylinder describ'd by the Rectangle $BDbd$. Since therefore $A : a :: BCq \times Bb + q : BDq \times Bb$, in finite Terms ; that is, as $BCq + \frac{q}{Bb} : BDq$, then

shall ultimate Ratio $\frac{A}{a} = \frac{BCq}{BDq}$. For when the

Points C and c coincide, the little Solid q describ'd by the Curvilinear Triangle CcE , vanishes. Therefore the Fluxion of the Solid describ'd by ABC , is to the Fluxion of the Solid describ'd by $ABGD$, as the Square of BC , to the Square of BD . $Q : E : I$.

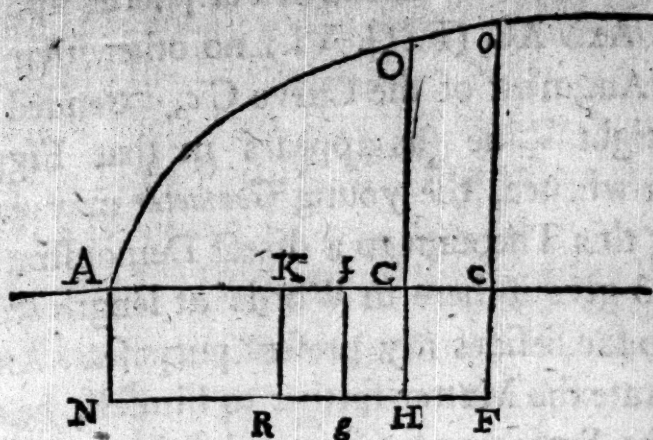
5. Lastly, Let it be required to find the proportion of the Fluxion of the Curve *Surface* generated by the Rotation of the Curve Line AC , to the Fluxion of the Cylindrick Surface describ'd by the right Line GD , revolving about the same Axis with the former, viz. AB . Now I believe it may be an easie and natural way of explaining this, to reduce it from a Curve *Surface*, to a Curvilinear *Area* ; for since we have

D 2

before

before demonstrated the Proportion between the Fluxions of a Curvilinear *Area* and a *Rectangle*; if we shew that the Curve *Surface* and the Cylinderick *Surface* are to be reduced to a Curvilinear *Area* and a *Rectangle*; then the Proportion of their Fluxions will be stated and determined by the former Principle. Now this is a most demonstrable Truth, that any *Surface* whether Rectilineal or Curvilinear, is equal to an *Area*, the Axis of which *Area*, if it be a rectilineal *Surface*, is the generating right Line, and if it be a Curve *Surface*, is the generating Curve Line extended into a right Line; and the Ordinates of which *Area*, are the Peripheries (the Radii of which are the respective Ordinates in the generating Figure) extended also into right Lines, and applied at right Angles to the fore-mentioned Axis. Ex. gr. in the Curve at FIG. V. suppose it to revolve about the Axe AB, by which revolution of the Curve Line ACc, the Curve *Surface* is generated. Now if we imagine the Curve AC (FIG.V.) to be extended into the right Line AC (FIG. VI.)

and



and the Peripheries, whose Radii are the ordinates BC , bc , &c. (FIG. V.) to be extended into the right Lines Co , co , &c. (FIG. VI.) and applied as ordinates to the Axis AC ; by this means there will be form'd a new Curve AOo , the *Area* of which, viz. $Ac o O A$, is equal to the Curve Surface describ'd by the revolution of the Curve Line AC (at FIG. V.) about the Axis AB . Neither can there be much difficulty in apprehending this. For as the Peripheries belonging to the Radii BC , bc , &c. (FIG. V.) lie parallel to each other upon the Curve Surface in their Circular Form; so the Peripheries, only extended into the right Lines CO , co , &c. (FIG. VI.) lie parallel to each other, ordinately applied to the Axis AC . And as the interval of any two Peripheries upon the convex Surface (FIG. V.) is the Augment of the Curve Line Cc lying there in the form of a Curve Line, so is the interval or distance,

stance of any two extended Peripheries applied
 to the Axis AC (FIG. VI.) no other than the
 same Augment of the Curve Cc, extended in-
 to a right Line (as appears in that Figure.)
 From whence, the young *Geometer* may easily
 bring this Theorem to a direct Demonstration;
 tho' to give it here in Words at length would
 be too far besides my present purpose. And to
 illustrate the Matter farther to himself, he may
 try the Surfaces of *Cones* and *Spheres* (which I
 suppose him to know how to find, from some
 other Principle) and see whether this Theorem
 does not give him the same Conclusions. This
 premised, let AK (in FIG. VI.) be taken equal
 to AB (in FIG. V. and $Kf = Bb$, and AN
 = the Periphery describ'd with the Radius bd
 or BD, and let the Parallelogram ANcF be
 completed. Then 'tis evident, that as the
 Area Aco (FIG. VI.) is equal to the Curve
 Surface generated by the Revolution of the
 Curve Line ACc (FIG. V.) so the Rectangle
 ANfg (FIG. VI.) is equal to the Cylindrick
 Surface generated by the Revolution of the Right
 Line G (FIG. V.) because the Lines KR and
 fg, &c. are by supposition equal to the Peri-
 pheries describ'd by the Lines BD and bd , &c.
 in their Revolution about the Axe Ab. So that
 now 'tis manifest we have nothing to do, but
 to find the proportion between the Fluxions of
 the Area ACO, and the Rectangle AR; for
 the

the former is equal to the Curve Surface, and the latter to the Cylindrick. Now the Augments (of these two flowing Spaces) generated in the same Particle of Time, are the Curvilinear Trapezium $COoc$, and the Rectangle $KfRg$; and the *last Ratio* of these, is the Ratio of the Fluxions. Now ult. Ra. $\frac{COoc}{KfRg} =$ ult.

$$\text{ra. } \frac{COoc}{CcHF} \times \text{ult. ra. } \frac{CcHF}{KfRg} \text{ (for if } \frac{COoc}{KfRg} \text{ be =}$$

$$\frac{COoc}{CcHF} \times \frac{CcHF}{KfRg}, \text{ as it evidently is; then also}$$

$$\text{will ult. ra. } \frac{COoc}{KfRg} = \text{ult. ra. } \frac{COoc}{CcHF} \times \text{ult. ra.}$$

$$\frac{CcHF}{KfRg}.) \text{ But ult. ra. } \frac{COoc}{CcHF} = \frac{OC}{HC} \text{ (as appears}$$

from what was said before about the Proportion of the Fluxions of a Curvilinear Area and a

$$\text{Rectangle) and ult. ra. } \frac{CcHF}{KfRg} = \text{ult. ra. } \frac{Cc}{Kf}$$

$$= \frac{\text{Fluxion of the Curve}}{\text{Fluxion of the Abscisse}} \text{ (which is manifest al-}$$

so from what has been shewn of the Ratio of the Fluxions of the Curve and the Abscisse; for

Cc and Kf at *Fig. 6.* are the same with Cc and Bb , or CE at *Fig. 5.*) Therefore ult. ra.

$$\frac{COoc}{KfRg} = \frac{OC}{HC} \times \frac{\text{Fluxion of the Curve}}{\text{Fluxion of the Abscisse}}$$

$$= \frac{OC \times \text{F. Curve.}}{HC \times \text{F. Abscisse.}} = \frac{OC \times \text{F. Curve.}}{Fg \times \text{F. Abscisse.}} \text{ But}$$

OC is equal Periph. with Radius BC (FIG. V.) and $fg =$ Periph. with Radius bd . Therefore (FIG. V.) the Fluxion of the Curve Surface generated by the Curve Line AC, is to the Fluxion of the Cylindrick Surface generated by the Right Line GD, as the Rectangle under the Periphery (describ'd with the Radius BC) and the Fluxion of the Curve Line, to the Rectangle under the Periphery (describ'd with the Radius bd) and the Fluxion of the Abscisse. Q: E: I.

S C H O L.

The Fluxions of Curvilinear Areas, Solids, and Surfaces, have hitherto been compared with the Fluxions of Rectangles, Cylinders, and Cylindrick Surfaces; and determined in that Relation. We have not said *absolutely*, the Fluxion of the Area or Solid, or Surface, is thus or thus; but the Fluxion of the Curvilinear Area, or Solid, or Surface, is to that of the Rectangle or Cylindrick Solid, or Surface, in such or such a Proportion. But now, because the Quantities that express these Proportions, on the part of the Rectangles, Cylindrick Solids, and Surfaces, are *permanent* or *standing* Quantities; it follows, that we may state the Fluxion of the Curvilinear Area, Solid, or Surface, *out of, or without that Relation,*

or, and say *absolutely*, that the Fluxion shall be expounded thus or thus, or the Fluxions shall ever be in *this* or *that* Proportion to another. Thus *Ex. gr.* 'twas shewn (at *Art. 3.*) that the Fluxion of the Area, was to that of the Rectangle, as $BC : BD$. But because of the suppos'd uniform Motion of the Line BD , along the Line AB , the Fluxion of the Rectangle will be *constant* or ever the same, also BD is constant; therefore the Fluxion of the Area, will be *as* BC the Ordinate, and the Fluxions of the Area will be every where, *as* the Ordinates respectively. In like manner, the Fluxion of the Solid (*Art. 4.*) will be ever *as* BC^2 , that is, *as* the Square of the Ordinate; and the Fluxion of the Curve, Surface (*Art. 5.*) will be ever, *as* the Rectangle under the Periphery describ'd with the Radius BC , and the Fluxion of the Curve Line. These Quantities are not in reality the Fluxions of the Area, Solid, or Surface, but they are *as these Fluxions*, and do serve to expound them, and in Use and Practice are to be taken for them. Thus also in the more general and abstracted (*Algebraical*) expressions of Quantity, if we suppose *Ex. gr.* the Quantity x to flow *uniformly*, and would find what Term or Expression will expound the Fluxion of the Quantity x^n . Let the Augment of the Quantity x , generated in a given Particle

Particle of time, be expressed by the Symbol o . Now in the same time that x flows into (or by flowing becomes) $x+o$, the Quantity

x becomes $x+o$; and from the method of infinite Series, $x+o$ is $= x \sqrt[n]{n o x} + \frac{n n - 1}{2} o o x$

$+ \frac{3}{6} n - 3 n + 2 n o o x$, &c. The Augments there-

fore of these flowing Quantities, generated in the same Particle of Time, are o , and

$\frac{n-1}{2} o x + \frac{n-2}{6} o o x + \frac{n-3}{6} o o o x$, &c.

the former being the Augment of x , and the latter

that of x ; and these are manifestly one to ano-

ther as 1 to $n x + \frac{n-1}{2} n o x + \frac{n-2}{6} n - 3 n + 2 n o x$, &c.

And the last Ratio of them (making o to

vanish) will be that of 1 to $n x$ for all the

Terms multiplied into o do now vanish also

So that the Fluxion of x , is to the Fluxion of

x , as 1 to $n x$ and consequently the Fluxion

of x being constant, or always as 1, the Fluxi-

on of x shall be expounded by $n x$ And let

it be observed once for all, that when at any

time

time hereafter, I say such or such an Expression is the Fluxion of such or such a flowing Quantity, I always understand it in that Sence that I have explain'd my self in, in this *Scholium*, and desire so to be understood by others.

SECTION III.

Of the Notation of Fluxions.

AND thus much may suffice to be said concerning the first and main Foundations of our Work. It may be proper now to advance something farther, and come to the *Operations* in the method of *Fluxions*.

A R T. I.

Indeterminate or *Flowing Quantities* we chuse generally to express by these Symbols, v, x, y, z , the latter Letters of the Alphabet; and *constant* or *determinate* ones, by a, b, c, d , &c. unless where express caution is given to the contrary. For this matter of the *Designation* of the Symbols, being perfectly arbitrary, any Symbols may represent flowing, or constant

Quan-

Quantities; only use and custom have hitherto established this sort of Notation.

The *Fluxions* (that is, the *Celerities of the Increase or Decrease*) of these flowing Quantities, are express'd by the same Letters respectively,

mark'd with a *Tittle* over Head; Thus, $\dot{v}, \dot{x}, \dot{y}, \dot{z}$, denote the Fluxions of v, x, y, z , and so of any other simple Integer Quantities. The Fluxions of *Fractions* are denoted by a *Point* or *Tittle* in the *Break* of the Line, that is drawn between the Numerator and Denominator. Thus

the Fluxion of $\frac{x}{y}$ is denoted by $-\frac{\dot{x}}{\dot{y}}$; the

Fluxion of $\frac{az + z^2}{a - z}$, by $\frac{az + z^2}{a - z}$; and in like manner for others.

A R T. II.

The Notation hitherto described relates to the *first* Fluxions of any flowing Quantities. The Fluxions of the *first* Fluxions of any flowing Quantities, are call'd the *second* Fluxions of those flowing Quantities; the Fluxions of the *second* Fluxions, are the *third* Fluxions of the flowing Quantities, and so on. Second, third and fourth, &c. Fluxions are express'd by the same Letters that denote the primary flowing Quantities, mark'd with 2, 3, or 4, &c.

Points

Points or *Tittles* over-head. Thus $\ddot{v}, \ddot{x}, \ddot{y}, \ddot{z}$ are the second Fluxions of v, x, y, z , and the first Fluxions of $\dot{v}, \dot{x}, \dot{y}, \dot{z}$. And $\dddot{v}, \dddot{x}, \dddot{y}, \dddot{z}$ are the third Fluxions of v, x, y, z ; the second Fluxions of $\dot{v}, \dot{x}, \dot{y}, \dot{z}$, and the first Fluxions of $\ddot{v}, \ddot{x}, \ddot{y}, \ddot{z}$. And $\ddddot{v}, \ddddot{x}, \ddddot{y}, \ddddot{z}$ are the fourth Fluxions of v, x, y, z ; the third of $\dot{v}, \dot{x}, \dot{y}, \dot{z}$, the second of $\ddot{v}, \ddot{x}, \ddot{y}, \ddot{z}$, and the first of $\dddot{v}, \dddot{x}, \dddot{y}, \dddot{z}$.

A R T. III.

The second, third, fourth, &c. Fluxions of *Fractions*, are denoted also by 2, 3, or 4, *Fluxions* Points, but placed Horizontally in the *Break* of the Line that parts the Numerator and De-

nominator. Thus, $\frac{az + z^2}{a - z}$, and $\frac{az + z^2}{a - z}$, and

$\frac{az + z^2}{a - z}$, &c. are the second, third, fourth, &c.

Fluxions of the Quantity $\frac{az + z^2}{a - z}$. If these

Fractions were Roots or Surd Quantities, the Fluxions of them would be denoted, after the same manner. And if instead of Fractions, we had compound Integer Surd Quantities, their

their Fluxions would be express'd by such a number of *Points* in the *Break* of the Line drawn over Head. As if the Quantity were $\sqrt{az---zz}$, the first, second, third, fourth, &c. Fluxions of it would be denoted by $\sqrt{az---zz}$, $\sqrt{az---zz}$, $\sqrt{az---zz}$, $\sqrt{az---zz}$, &c.

A R T. IV.

But this Notion of Fluxions and flowing Quantities, may be extended yet a vast way farther. For all those Quantities that we vulgarly call *Flowing ones*, may themselves be consider'd as the Fluxions of others ; and them again, as the Fluxions of others, and so the Progression to be as boundless this way, as it is the other way. The Quantities in these Orders (*going upwards* beyond the vulgar Flowing Quantities) are denoted by the same Letters with the vulgar ones, only mark'd with one, two, three, &c. little *stroaks* or *dashes*, over-head : As the Quantities in the other Orders, *going downwards* are denoted by the same Letters, with the like number of *Points*.

Thus as $\ddot{z}$, $\dot{y}$, $\dot{x}$, v , are the Fluxions of $\ddot{z}$, $\dot{y}$, $\dot{x}$, v , and these the Fluxions of $\ddot{z}$, $\dot{y}$, $\dot{x}$, v ; and these the Fluxions of $\ddot{z}$, $\dot{y}$, $\dot{x}$, v ; so the Quantities $\ddot{z}$, $\dot{y}$, $\dot{x}$, v , may be consider'd as the Fluxions of

of z, y, x, v ; and these as the Fluxions of

z, y, x, v ; and these as the Fluxions of $z, y,$

x, v , and so *ad Infinitum*. And the Fluxions of

broken or surd, or compound Quantities are in like manner express'd in this kind of Notation;

the Dash, or Dashes being placed just over

the *Break* of the Line, as in the other Case, the

Point or *Points* were placed in it. *Ex. gr.*

$\sqrt{ax-zz}$, is the Fluxion of $\sqrt{ax-zz}$, as this

latter is of $\sqrt{ax-zz}$; and so in the rest. Now

then in such a Series as this, $z, z, z, z, z, z,$

$z, \&c.$ or in such a Series as this, $\sqrt{ax-zz},$

$\sqrt{ax-zz}, \sqrt{ax-zz}, \sqrt{ax-zz}, \sqrt{ax-zz}, \sqrt{ax-zz},$

$\&c.$ continu'd both ways as far as we please;

every Quantity is the Fluxion of the immedi-

ately foregoing, and the Fluent of the imme-

diately following one; and comparing them

at any Intervals or Distances, they are Fluxions

or Fluents of such a *Denomination*, first, second,

third, $\&c.$ according to the number of Inter-

vals between the Terms compar'd.

SECT.

SECTION IV.

Of the Operations in Fluxions.

THese Things premised, we come now to propose a General Problem; the Solution of which, is the Foundation of all our Operations about Fluxions.

P R O B.

An Equation being given, including any Number of Flowing Quantities; 'tis required to find the Fluxions of the same.

For the Solution of which, Mr. Newton gives this Rule, viz. Let each Term of the Equation be multiplied by the Exponent of the Power of every flowing Quantity, that Term includes, and in the several Multiplications, instead of the Root of any Power of each flowing Quantity, substitute its Fluxion; then will the Sum of all those Products under their proper Signs, be the new Equation sought. Ex: gr: let the Equation $x^3 - xyy + aaz - b^3 = 0$ be propos'd; where a and b are standing Quantities, x , y , z , flowing ones. Now let the Terms be multiplied first of all,

by

by the *Indexes*, of the Powers of x which they contain; then since x is not at all in any but the two first Terms of the Equation, if we multiply them Terms by the exponent of the Powers of x , which are 3 and 1, we shall have

$3x^3 \rightarrow xyy$; and putting in $\dot{x}$ (Fluxion) in the room of the Root of those Powers of x (that is, instead of x it self; or x of one dimension)

we shall then have $3x^2 \dot{x} \rightarrow xyy$. If we proceed after the same manner with the Quantity y , the

result of that will be $\rightarrow 2xy\dot{y}$; and if with the

Quantity z , we shall have $a\dot{a}z$. So that the Sum of all the Products with their respective

Signs, is $3x^2 \dot{x} - yy\dot{x} - 2xy\dot{y} + a\dot{a}z$. Put this Expression $= 0$, and then will this Equation give us the relation of the Fluxions, of the flowing Quantities (as so combin'd and related, in the Equation propos'd) or in other Words, this Equation will be the Fluxion of the other. The Demonstration of which is as follows.

A R T. III.

Let the very small Quantity o be taken, and let the Expressions $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, represent the
E (Moments)

(*Moments*, or, *Increments* (of those flowing Quantities, z, y, x , respectively) generated in a very small part of Time. If therefore now, at the present *Moment*, the flowing Quantities are z, y, x ; the next *Moment*, (when augmented by these *Increments*) they will become

$z + oz, y + oy, x + ox$. Substitute these Expressions (in the Equation propos'd) in the room of z, y and x , respectively; and then

that Equation will appear thus, viz. $x^3 + 3x^2$

$$x + 3xox + ox^2 - xy^2 - oxy^2 - 2xoyy -$$

$$2xoyy - xoyy - xoyy + az + aoz -$$

$= 0$. The Equation propos'd being compr-

hended in this, subtract that from this, and

divide the remainder by the quantity o , and

$$\text{then it will be } 3xx + 3xox + ox^2 - yx^2 -$$

$$2xyy - 2oyyx - xoyy - xoyy + aaz = 0$$

Let the Quantity o be diminished Infinitely

or supposed to vanish, and then all the Term

into which it is multiplied, vanishing also,

$$\text{there will remain } 3xx - yx^2 - 2xyy + aaz$$

$= 0$, the Equation that determines the Relation

of the Fluxions.

SCHOL

S C H O L. I

This Process is in the main, exactly the same with that which has been us'd so often before, for determining the Proportions of Fluxions. And to make it as plain as possible that 'tis so, and to shew, that to work by this Rule, here delivered and demonstrated, and to work by that delivered at *Art. 4. Pag. 5.* is one and the same thing in effect, we will take an Equation, and work it according to the directions of both Rules, and so compare the results on both sides. Let the Equation be $aa\dot{z} - x\dot{y}y = 0$, or $aa\dot{z} = x\dot{y}y$. Then, according to the Rule in this Problem, the Equation expressing the relation of the Fluxions, will be $aa\dot{z} - y\dot{y}x - 2y\dot{y}x = 0$; the Operation for which I need not repeat. Now according to *ART. IV. Pag. 5.* if we take the last Ratio of the Increments generated in a given Particle of Time, that will give us, the Proportion and Relation, of the Fluxions of any fluent Quantities propos'd. Now to state the *Increments* rightly and congruously; we are to consider that here being several flowing Quantities, x, y, z ; which are to be supposed to flow variously; the same small Quantity o cannot possibly represent their several Increments generated in

the same Particle of Time: for they cannot be all of the same Magnitude. So that there must be a different *Expression of Increment*, peculiar to and each of the flowing Quantities, x, y, z ; be these Expressions are very aptly made to $\dot{o}x, \dot{o}y, \dot{o}z$; concerning the reason of which we shall speak farther by and by. In the same Moment of Time therefore, the Quantities x, y, z , flow into $x + \dot{o}x, y + \dot{o}y, z + \dot{o}z$; and in the same Moment that y becomes $y + \dot{o}y$, yy becomes $yy + 2\dot{o}yy + \ddot{o}yy$; and in the same Moment that yy becomes $yy + 2\dot{o}yy + \ddot{o}yy$, the Quantity xyy becomes $xyy + 2xy\dot{o}y + x\ddot{o}yy + \dot{o}yyx + 2y\dot{o}oyx + \overset{3..2}{\ddot{o}oxy}$, and the Quantity aaz , becomes $aaz + a\dot{a}oz$. Therefore the Augments of xyy , and aaz , generated in the same Moment, are $2xy\dot{o}y + x\ddot{o}yy + \dot{o}yyx + 2y\dot{o}oyx + \overset{3..2}{\ddot{o}oxy}$, and $a\dot{a}oz$; and these are evidently one to another as $2xy\dot{o}y + x\ddot{o}yy + \dot{o}yyx + 2y\dot{o}oyx + \overset{2..2}{\ddot{o}oxy}$, to $a\dot{a}z$. And for the last Ratio of them; let the augmenting Quantity of o vanish; and then they will be as $2xy\dot{y} + yy\dot{x}$, to $a\dot{a}z$, which is there-

therefore the Ratio or Relation of the Fluxions desired. But now, because the flowing Quantities are equal, viz. $xyy = aaz$, therefore their Increments generated in a given Particle of time, are equal; that is, the Ratio of those Increments consider'd as *finite*, is a Ratio of Equality; and therefore the Ratio of those Increments consider'd as *nascentia*, or *evanescentia*, shall be a Ratio of Equality; and therefore the Ratio of the Fluxions shall be a Ratio of Equality; that is, $2xy\dot{y} + yy\dot{x} = a\dot{a}z$; and consequently $a\dot{a}z - yy\dot{x} - 2xy\dot{y} = 0$, which is the Equation found by the former Rule.

N. B. I inferr'd just now, that if the Ratio of the *finite* Increments was a Ratio of Equality, that of the *nascentia*, or *evanescentia* Increments, should also be a Ratio of Equality. I would not be thought to suppose by this, that the Ratio of the *nascentia*, or *evanescentia* Increments shall be the same with that of the Increments consider'd as *finite*; for we have sufficiently shewn the contrary to that already. But what I assert is, That if the Increments consider'd as *finite* and actually generated, in any given Particle of time, be equal to one another, that then they are also equal in the *first*, *arising*, or *last*, *vanishing*, state; or that a Ratio of Equality between *finite* Increments, generated in the same time, can never become a Ratio of Inequality.

lity, in Increments consider'd as *arising*, or *vanishing*.

SCHOL. II.

The Augments of the flowing Quantities (z , x , y ,) generated in the same Particle of time, and represented by the Expressions oz , ox , oy ; are by a very proper and significant Name, called *Moments*. This Term *Moment*, we know, in its common use, intimates, a Product of Magnitude into Velocity; and it appears to do the same here. For oz *ex. gr.* is the Product of o multiplied into z ; that is, of the small Quantity o multiplied into z , the Fluxion or Velocity of the *arising* Increment, of the flowing Quantity z ; and so of the rest. And this sort of Notation seems to be so natural, and apt to the purpose, that 'tis not easie to imagine, what other Expression can be found, that it should give way to.

For though the Increment of any one flowing Quantity, as z , or y , or x , may congruously enough be represented by one small Quantity, as o ; yet the Increments of several together can't possibly be express'd by one and the same Quantity, because the *Lex & ratio Fluendi*, is not the same in all of them, but *different and various*.

various. Therefore we can't put $z+o$, $y+o$, $x+o$, for the Increments of z , y , x , generated in a given Moment of time. On the other hand, neither can we represent the Increments of these flowing Quantities thus, viz, $z+z$, $y+y$, $x+x$. For $\dot{z}$, $\dot{y}$, $\dot{x}$, denote the Fluxions, that is, the Velocities of the arising Increments of these flowing Quantities; and 'twould be disproportional to represent the *Incrementa* of any Quantities, by Symbols that denote meer Velocity: So that neither of these two Expressions can do alone and by themselves. But now we may fairly represent these Increments, by the Products $o\dot{z}$, $o\dot{y}$, $o\dot{x}$. For here is *difference* and *variety* in this Notation, that serves to express the *different* Increments of *different*, and *differently flowing*, Quantities; because tho' one *Factor*, namely o , is the same in all, yet the other *Factors* are all different, and consequently the whole *Momenta* are so too. But then (which is the main Point) these *Momenta* are in proportion to one another, as the Fluxions of the flowing Quantities respectively, for $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, are as $\dot{z}$, $\dot{y}$, $\dot{x}$; and Mr. Newton had before expressly told us, that the Increments generated in a very small Particle of time were *very nearly*, as the Fluxions. So that in congruity to that Fundamental Law, the Incre-

ments ought to be expressed so, as that their Expressions might be proportional to those of the Fluxions ; and this could not be, but by representing of them in this manner as *Momenta* or Products. From whence the considerate Reader may see what wonderful Art and Contrivance there is, even in the most minute Steps taken by the great Author in this Method.

Thus have we given the Fundamental Demonstrations, of all our future Operations, in finding the Fluxions of any flowing Quantities. What remains now, is to bring the Matter down to easie and common practise, in all variety of Cases that can happen ; and to shew in a *demonstrative* way, how those Rules are formd, which the young Geometer is to make use of, for speedy and expeditious performance. And all these things will be but so many Examples and particular Applications of the General Rule before given ; for that contains all : However, it will be very useful thus to illustrate it, and branch it out into those particular Rules, that must serve for practice.

A R T. IV.

Whatever flowing Quantity is proposed, 'twill be reduced to one of these Heads: An *Aggregate* (*Sum* or *Difference*, of several flowing

ing Quantities) a *Product*, or Multiplication of several flowing Quantities into one another : A *Fraction* or Quotient of some flowing Quantities divided by others : A *Power* of some flowing Quantities (whether a *simple Quantity*, a *Product*, a *Quote* or *Fraction*.) As for *Roots* and *surd Quantities*, I include them under the Article of *Powers*. And I forbear making a distinct Article of *Aggregates* of *Products*, *Quotes*, or *Powers*, because the Rules for them single, will reach all possible *Combinations* of them.

To proceed then. Let the *Aggregate* $x + y + z$ be propos'd. The Fluxion then will stand thus, $\dot{x} + \dot{y} + \dot{z}$. And so in other Instances of the like Nature ; neither need this case be insisted on. The Rule here, *is only* (according to the Notation) *to set down the flowing Quantities, with the Points over them each with their respective Signs.*

A R T. V.

Let the *Product* $x y z$ be propos'd. Suppose $x y z = v$; then will the Fluxion of v ; that is $\dot{v}$, be equal to the Fluxion of the quantity propos'd ; *which I desire should always be observed.* Let the *Momenta* (or *Increments* of these flowing Quantities, generated in a given Particle

Particle of time) be oz , ox , oy , ov , the reason of which was shewn before. Then substituting $z + oz$, $y + oy$, $x + ox$, $v + ov$, instead of z , y , x , v ; we shall have for a new Equation $zxy + yzox + zxoy + yxoz + zox\dot{y} + yooz\dot{x} + xooz\dot{y} + ozy\dot{x} = v + ov$. From whence taking away the Original Equation $zxy = v$; there remains $yzox + zxoy + yox\dot{z} + zox\dot{y} + yooz\dot{x} + xooz\dot{y} + ozy\dot{x} + ozx\dot{y} = ov$; that is, $yzx + zxy + yxz + zox\dot{y} + yox\dot{z} + xooz\dot{y} + oozy\dot{x} = v$; and making the Quantity o vanish, we have $yzx + zxy + yxz = (v =)$ the Fluxion of the Quantity propos'd. So that the Rule is, *to multiply the Fluxion of every particular flowing Quantity, by the Product of the rest of the flowing Quantities, and the sum of all those Products is the Fluxion required.* As here x is multiplied by yz , y by zx , z by yx , and the sum of these three Products, is the Fluxion of zyx .

So the Fluxion of axy , is $axy + yax$, for a being a constant Quantity, the Fluxion of it is equal to nothing, and so the Term in which
that

that would be found, is equal to nothing ; *which I remark here once for all.*

So the Fluxion of $ax + bx + cy$, is $\dot{a}x + \dot{b}x + \dot{c}y$. And the Fluxion of $xyzv$, is equal $\dot{x}yzv + y\dot{x}zv + z\dot{y}xv + v\dot{x}yz$.

A R T. VI.

Let the Fraction $\frac{x}{y}$ be propos'd. Suppose $\frac{x}{y} = v$, or $x = yv$. Then $x + o\dot{x} = yv + v\dot{o}y + yo\dot{v} + o\dot{o}y\dot{v}$, and (because $x = yv$) $o\dot{x} = v\dot{o}y + yo\dot{v} + o\dot{o}y\dot{v}$; that is, $\dot{x} = v\dot{y} + y\dot{v} + o\dot{o}y\dot{v}$; and when o vanishes, 'tis $\dot{x} = v\dot{y} + y\dot{v}$, and by reduction $\dot{v} = \frac{\dot{x} - v\dot{y}}{y}$. In this expression, instead of v , substitute its value $\frac{x}{y}$, and then $\dot{v} = \frac{\dot{x} - \frac{x\dot{y}}{y}}{y} = \frac{y\dot{x} - x\dot{y}}{yy}$, which

is the Fluxion desir'd. The Rule then is this, *Multiply the Fluxion of the Numerator, by the Denominator, and after it place (with the Sign —) the Fluxion of the Denominator, multiplied into the Numerator; then divide the whole by the Square of the Denominator, and this Fraction is the Fluxion*

Fluxion of the Fraction propos'd. As here, $\dot{x}$ (the Fluxion of the Num. is multiplied by y the Denominator : After which is placed (with the Sign—) $\dot{y}$ the Fluxion of the Denom. multiplied into x the Numer. and all is divided by yy the Square of the Denom. y . So the Fluxi-

on of $\frac{1}{z}$, is $-\frac{\dot{z}}{zz}$; of $\frac{a}{z}$, is $-\frac{a\dot{z}}{zz}$; for here the Numer. is a standing Quantity. The

Fluxion of $\frac{x}{a+x}$, is $= \frac{ax}{aa + 2ax + xx}$; and

the Fluxion of $\frac{xy}{x+a}$, is $\frac{\quad}{\quad}$

$$\frac{axy + ayx + xxy + yxx - xyx}{aa + 2ax + xx} =$$

$\frac{axy + ayx + xxy}{a^2 + 2ax + x^2}$ as is plain; for the Fluxion of the Numer. xy , is $x\dot{y} + y\dot{x}$, which multiplied by the Denom. $a+x$, gives $axy + ayx + xxy + yxx$, from whence subtracting the Fluxion of the Denom. (which is $\dot{x}$) multiplied into the Numerator xy , which makes yxx , and rejecting contradictory Terms; and lastly dividing all by the Square of $a+x$ the Denominator, the Fluxion will stand in that Posture as

is

is express'd. This the Reader may demonstrate to himself by supposing $\frac{xy}{a+x} = v$, and so finding the value of v , by just Transposition and Reduction as before.

A R T. VII.

In order to the right understanding and management of the Operations for the Fluxions of Powers, 'twill be requisite to premise some few general things, concerning the nature of Powers themselves, and their Exponents. And here I think it may not be an improper Description, to say, *That a Power is the Result or Product, of a certain number of Multiplications, where the Multiplier is the same Quantity continually.* And if we make Unity to be the first Multiplicand, then the Power produced, will be denominated, the *first, second, third, fourth,* &c. Power, according to the number, either of Multiplications, of Multiplicands, or Multipliers. Suppose any Quantiy, as q ; and then

will this Product, viz. $1 \times q \times q \times q \times q \times q \times q$, &c. either continued, or contracted, according to any Number of Multiplications (more or fewer)

fewer) that we have a mind to make, be a certain Power of the Quantity q , denominated and express'd, by that number, which is the number of the Multiplications that are made, or of the Multipliers, or Multiplicands (to be consider'd in those Multiplications) *Ex. gr.* suppose we should stop at three Multiplications, the



Result or Product would be $1 \times q \times q \times q$, or $q \times q \times q$, or $q q q$, that is the third Power of q . That there are three Multiplications here (and no more than three) is evident; for first, here's the Multiplication of 1 by q ; next the Multiplication of 1 into q , by q ; and then the Multiplication of 1 into q , into q , by q . The three Multipliers also, are q , q , and q ; and the three Multiplicands, are 1, $1 \times q$, and $1 \times q \times q$. And so in like manner for any other Power; the number that denominates or expresses the *Rank or Place* of that Power, in the order of first, second, third, fourth, &c. is the number that shews or expresses how many Multiplications are made, *in order to the Generation of that Product or Power.*

A R T. VIII.

These Numbers now, that denominate the *Rank or Place* of any Powers, in the order of

Nume-

Numeration, are very properly stil'd, their *Indices* or *Exponents*. This Name is suited to their Nature and Office, which is to indicate, expound, or shew, how those Powers stand, or are placed, in the order beforemention'd, and what Denomination they have there. So that taking the Powers in the natural and orderly Progression of first, second, third, fourth, fifth, &c. their Indices or Exponents, will consequently be, the natural Numbers beginning from Unity, viz, 1, 2, 3, 4, 5, &c. From hence now follow those two Points, which are Fundamental in this Business.

1. That a Series of the Powers of any Quantity, thus form'd by continual Multiplications, are a Series of continued Geometrical Proportionals.

2. That the Indices or Exponents of these Powers, are a Series of continued Arithmetical Proportionals.

Ex. gr. suppose

$1 \times y \times y \times y \times y \times y \times y \times y$, &c

and setting down the several Products (taken according to the extent of the Lines drawn over Head) as distinct Quantities by themselves; we shall have a Series of the Powers of y , viz.

y .

y . yy . yyy . $yyyy$; that is, the first, second, third, &c. Powers of the Quantity y ; which are called by the names of the *Side*, or *Root*, *Square*, *Cube*, *Biquadrate*, &c. and are very neatly and significantly represented by the same Letter y , with the number that denotes the Exponent of each Power, plac'd at the Head of it, as

$^1 y$ $^2 y$ $^3 y$ $^4 y$ $^5 y$ &c.

Now that these are a Series of continued Geometricals, is most evident. For the Ratio of any two Terms immediately following one another, is ever that of 1 to y , as appears from the Law of the Generation or Formation of those Powers; in which the Multiplication proceeds continually by the same Quantity y . And that the Exponents are a Series of continued Arithmetical Proportionals, is as plain; for they go on, having still the same excess or difference between one another; as $2 - 1 = 1$, $3 - 2 = 1$, $4 - 3 = 1$, and so in the rest. These things are clear from the very Definitions of Geometrical and Arithmetical Proportion.

A R T. IX.

And from hence it will follow, that if the first Term of the Geometrick Series, be made 1; the first Term of the Arithmetick Series must be 0; because as 1. y . y^2 , are in continual
Geo-

Geometrical, so 0. 1. 2, are in continual Arithmetical Proportion. So that then they

will stand thus, viz. $\begin{matrix} 2 & 3 & 4 & 5 \\ 1. & y. & y. & y. & y. & y. & \&c. \end{matrix}$ From
 $\begin{matrix} 0. & 1. & 2. & 3. & 4. & 5. & \&c. \end{matrix}$

whence it will follow also, that 1, in the Geometri-

cal Series, is the same thing as y^0 , or the Quantity y , having a Cypher for its Exponent. For the Numbers in the Arithmetical Series, do (as was shewn) expound or express, the correspondent Powers, in the Geometrical Series. So

that as y^2 and y^1 , are the Terms corresponding to 2 and 1 (in the Arithmetical Series) so al-

so must y^0 be the Term, corresponding to 0 in

that same Series. For as y^2 is the result of two Multiplications, where y is the continual Mul-

tiplier, and y^1 or y (or $y \times 1$) is the result of

one such Multiplication; so is y^0 the result of no Multiplication at all, in which y is any Multiplier: or it is not produced by any Multiplication of y . Therefore since *from the nature of Geometrick Proportion*, 1 will be the next Term below y , and also since *from the na-*

ture of Powers, y^0 will be the next Term below y ; 'tis evident that these two Expressions are

equal, or that $1 = y^0$.

F

ART.

A R T. X.

From what has hitherto been said, may be deduced the two Fundamental Rules, for all Operations relating to Powers and Exponents.

1. That any two Terms in the Geometrick Series, being multiplied into each other, the *Product* will be a Term, the Exponent of which will be the Sum of the Exponents of the two other Terms added together. *Ex. gr.* I say,

that $y^2 \times y^6$ is $= y^{2+6} = y^8$, and $y^3 \times y^4$ is $= y^{3+4}$

$= y^7$; and so in any the like case. This may be shewn (in some sort) from the very Notation it self :

For y^2 is yy , and y^6 is $yyyyyy$, and

y^8 is $yyyyyyyyyy$, expanding the Powers from the *Numeral*, into the literal Expression. And that the two former Expressions multiplied, are equal to the latter, is evident to the sight. But the matter is capable of a fair Demonstration, from the Nature and Properties of Geometrick Proportion. For (because of the Terms in the same Proportion to one another) these Analogies hold good, *viz.*

⁵ ⁶ ⁷ ⁸ } For these Terms are in
 Tis $y : y :: y : y$ } Geom. Proportion.

⁶ ⁸ ⁵ ⁷
 Or $y : y :: y : y$ Inverting and Permuting:

⁵ ⁷ ⁴ ⁶
 But $y : y :: y : y$

⁴ ⁶ ³ ⁵ } By Inversion and Per-
 And $y : y :: y : y$ } mutation of propor-
³ ⁵ ² ⁴ } tionate Terms.
 And $y : y :: y : y$

² ⁴ ²
 And $y : y :: 1 : y$
⁶ ⁸ ²

Theref. $y : y :: 1 : y$ By Equality of Proport.

Now from the nature of Geometrical Proportion, the Product of the two Extrems, is equal to the Product of the two Means; there-

⁶ ² ⁸ ⁸ ⁶⁺²
 fore $y \times y$ is $= 1 \times y = y = y$. Thus by a *Process of Analogies*, may we bring the two Terms (which we suppose to be multiplied into one another, and the Product of which, we say, is a Term, whose Exponent is equal to the Sum of the other two Exponents) we may, I say, bring these two Terms to be the two *Extrems* of an Analogy, the two *Means* of which, shall be 1, and the Product of those two Terms. And thus may any other Examples be demonstrated.

A R T. XI.

And here, I think, will naturally come in, the consideration of Powers (whose Exponents are Negative, and) which compose the Reciprocal Series (as they are commonly call'd)

suppose the Series, $y^4 y^3 y^2 y^1 y^0$ were to be continued on from 1 or y , according to the nature and tenour of that Law, relating to the multiplication of Powers, just now demonstrated. The Question is, what will be the Exponents of the Powers of y in the Geometrick

Series continued from 1 or y . Let us represent those Exponents by a, b, c , &c. and so express the continuation of the Series, by the Terms

$a \ b \ c$
 $y^a y^b y^c$ &c. So that the Series will stand thus;

$4 \ 3 \ 2 \ 1 \ 0 \ a \ b \ c \ d$
 $y^4 y^3 y^2 y^1 y^0 y^a y^b y^c y^d$ &c. Now because the Terms are Geometrically proportional, there-

$2 \ 1 \ 0 \ a$
fore $y^2 : y^1 :: y^1 : y^a$, and the Product of the *Extremes* being equal to the Product of the *Means*,

$1 \ 0 \ 2 \ a$
it will be $y^1 \times y^0 = y^2 \times y^a$, or (because $y^0 = 1$)

$1 \ 1 \ 2 \ a$
 $y^1 \times 1$ or $y^1 = y^2 \times y^a$; and therefore according to the Rule, 1 must equal the Sum of the Exponents 2 and a ; that is $1 = 2 + a$, consequently

quently $1 - 2 = a$, that is $-1 = a$. After the same manner $b, c, d, \&c.$ will be found to be $-2, -3, -4, \&c.$ respectively. So that the Series continued will appear in this Form,

$\overset{4}{y} \overset{3}{y} \overset{2}{y} \overset{1}{y} \overset{0}{y} \overset{-1}{y} \overset{-2}{y} \overset{-3}{y} \overset{-4}{y} \&c.$ Farther,

I say, that $\overset{-1}{y} \overset{-2}{y} \overset{-3}{y} \&c.$ are the same with

$\frac{1}{y} \cdot \frac{1}{y^2} \cdot \frac{1}{y^3} \&c.$ So that the Series will be the same which way soever it is expressed. For 'twas shewn just now, that

$\overset{2}{y} \overset{1}{y} \overset{0}{y} \overset{-1}{y}$ are Geometrick Proportionals. But

so also are, $\overset{2}{y} \overset{1}{y} \overset{0}{y} \overset{1}{y}$, for by the Rule of Mul-

tiplication of Powers, $y^2 (= y^{1+1}) = y^1 \times y^1$, there-

fore $y^2 \times \frac{1}{y} = \frac{y \times y}{y^1} = y$. But we have shewn too

that $y^0 = 1$, consequently $y^0 \times y^1 = y$; from whence it follows, that the Product of the Ex-

trems is = the Product of the Means, and consequently, that $\overset{2}{y} \overset{1}{y} \overset{0}{y} \overset{1}{y}$ are Geom. Pro-

portional. From hence we may infer, that $y^{\overset{-1}{}}$

is the same thing as $\frac{1}{y}$, since they are either of them a fourth Proportional to these three, viz.

$y^2 \cdot y^1 \cdot y^0$ And so of any others. Therefore if we have any Quantities expressed in this Form, $\frac{1}{y} \cdot \frac{1}{y^2} \cdot \frac{1}{y^3} \cdot \&c.$ where the Exponents are positive, they may be changed into another Form, where the Exponents are Negative; and so become $y^{-1} \cdot y^{-2} \cdot y^{-3} \cdot \&c.$

A R T. X.

The second Rule to be observed is, *viz.* That taking any two Terms in the Geometrick Series, if one be divided by the other, the *Quotient* will be a Term, the Exponent of which will be equal to the Exponent of the Numerator less the Exponent of the Denominator. Thus

$\frac{y^3}{y^5}$ shall $= y^{-2}$, and $\frac{y^6}{y^3}$ shall $= y^3$, and $\frac{y^5}{y^3}$ shall be $= y^2$, or to $\frac{1}{y^{-2}}$. For (by the Rule of Multipli-

cation of Powers) $y^8 = y^6 \times y^2$, therefore $\frac{y^8}{y^5} = \frac{y^6 \times y^2}{y^5} = y^3$ So (for the same reason) $y^6 = y^3 \times y^3$, therefore $\frac{y^6}{y^3}$ shall $= \frac{y^3 \times y^3}{y^3} = y^3$

Now in the former Example, $2 = 8 - 6$, and in the latter, $3 = 6 - 3$; that is, the Index of the Quotient, is = the Index of the Denominator

nator subtracted from the Index of the Numerator. Again, $\frac{y^5}{y^8} = \frac{y^5}{y^6 \times y^2} = \frac{1}{y^2} = y^{-2}$, from what was just now demonstrated; but now -2 is $= 6 - 8$, so that the Exponent of the Quotient will still be the remainder, when the Exponent of the Denominator is subtracted from that of the Numerator.

A R T. XIII.

The Rules relating to the Exponents of the Square, Cube, &c. or of the Square, Cube, &c. Root, of any Term in the Geometrick Series are but particular Cases of these Rules for Multiplication and Division of Powers. Thus

the Square of y^3 , that is, the Product $y^3 \times y^3$, is

by the Rule of Multiplication, equal to y^6 ; and $6 = 2 \times 3$, so that the Exponent of the Square is double the Exponent of the Side. So the

Cube of y^2 , that is, the Product $y^2 \times y^2 \times y^2$, is by

the Rule of Multiplication, equal to y^6 , and $6 = 2 \times 3$, which shews, that the Exponent of the Cube, is tripple the Exponent of the Root or Side. And Universally, the Exponent of any Power, is so many times the Exponent of the Root, as is expressed by the number that

denominates that Power, in the order of first, second, third, &c. Or, the Exponent of the Power, is to the Exponent of the Root, as the number that denominates the Power, to Unity.

A R T. XIV.

Thus much for Powers, where the Exponents are *Integers*, either *positive* numbers, as

I. $y^1 y^2 y^3 y^4$ &c. or *Negative* Numbers, as
O. 1. 2. 3. 4. &c.

---1 ---2 ---3 ---4
I. $y^{-1} y^{-2} y^{-3} y^{-4}$ &c.
O. ---1. ---2. ---3. ---4. &c.

The *Roots* of Quantities are *Powers*, whose Indices or Exponents, are *Fractions*; or Powers, with Fractional Exponents, are such Roots of those Quantities, as the Fractions express. The Roots of Quantities, *ex. gr.* the Square, Cubick, Biquadratick, &c. Roots of y , may

be expressed thus, $\sqrt[2]{y}$, $\sqrt[3]{y}$, $\sqrt[4]{y}$, &c. And so the Square, Cubick, &c. Roots of any Power of y , as yyy , or y^3 may be expressed in this man-

ner, $\sqrt[2]{yyy}$, $\sqrt[3]{yyy}$, $\sqrt[4]{yyy}$, $\sqrt[5]{yyy}$, &c. But the following Notation is much more clean, natural and significant, *viz.* $y^{\frac{1}{2}}$, $y^{\frac{1}{3}}$, $y^{\frac{1}{4}}$, $y^{\frac{1}{5}}$, &c. for the
Square

Square, Cubick, Biquadratick, &c. Roots of y , and $y^{\frac{1}{2}}$, $y^{\frac{2}{3}}$, $y^{\frac{3}{4}}$, $y^{\frac{4}{5}}$, &c. for the Square, Cubick, Biquadratick, Surfolid, &c. Roots of yyy . In which Fractional Exponents, the Numerator ever denotes what power of the Quantity 'tis, the Root of which is intended to be taken; and the Denominator shews what that Root is, and how to be called, whether Square, Cubick, Biquadratick, &c. Root, according to the order of the Numbers, 2, 3, 4, &c.

The Origin and Generation of all these Fractional Powers, is in all respects like the former. Let the Quantity $\sqrt{y}$ or $y^{\frac{1}{2}}$ be propos'd, then will the Product

viz. $1 \times y^{\frac{1}{2}} \times y^{\frac{1}{2}} \times y^{\frac{1}{2}} \times y^{\frac{1}{2}} \times y^{\frac{1}{2}}$ either continued or contracted, be such or such a Power of $y^{\frac{1}{2}}$, according to such or such or such a number of Multiplications; the continual Multiplier being $y^{\frac{1}{2}}$. And taking the particular Products, limited by the Lines drawn over head, we shall have a Series of the Powers, of the Quantity $y^{\frac{1}{2}}$. *viz.* $1, y^{\frac{1}{2}}, y^{\frac{1}{2}} y^{\frac{1}{2}}, y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}}, y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}}, y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}} y^{\frac{1}{2}}, \&c.$ That is $y^0, y^{\frac{1}{2}}, y^1, y^{\frac{3}{2}}, y^2, y^{\frac{5}{2}}, \&c.$ the Exponents of which will be $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \&c.$ And if (as before)

fore) the Series were to be continued, the other way from y^0 or 1, it would be
 $y^{-\frac{1}{2}} \cdot y^{-1} \cdot y^{-\frac{3}{2}} \cdot y^{-2} \cdot y^{-\frac{5}{2}}$. or (which is all one)
 $1 \cdot \frac{1}{y^{\frac{1}{2}}} \cdot \frac{1}{y} \cdot \frac{1}{y^{\frac{3}{2}}} \cdot \frac{1}{y^2}$ &c. the Exponents of which therefore, are 0. — $\frac{1}{2}$. — 1. — $\frac{3}{2}$. — 2. — $\frac{5}{2}$. &c. The Reason and Demonstration of all which, is very evident from what has been already said about *Integer Powers*; the Rules demonstrated there, serving equally here; so that there needs be no Repetition made.

A R T. XV.

I shall only now make two or three General Reflections upon this business of Powers and their Exponents, and so proceed to the Application of them in Fluxions.

I. That, 'tis plain, that taking any Series of Quantities whatsoever in continued Arithmetick Proportion, we may presently form a Geometrick Series, to the several Terms of which the others shall be *Exponents* respectively. And *Vice versa*, a Series of Geometrick Powers, given, the Correspondent Arithmetick Series, is forthwith determin'd.

A R T.

A R T. XVI.

2. Taking any two Terms in the Geometrick Series; how many Geometrick mean Proportionals, there are included between these two Terms; so many Arithmetick *Means* there are, between the two Terms in the Arithmetick Series, that answer to the two Terms taken in the Geometrick Series. Thus $o \frac{1}{4} \cdot \frac{2}{4}$.

$\frac{3}{4} \cdot 1 \cdot \frac{5}{4} \cdot \frac{6}{4} \cdot \frac{7}{4} \cdot \&c.$ being Arithmetical Pro-

portional, and consequently $1 \cdot x^{\frac{1}{4}} \cdot x^{\frac{2}{4}} \cdot x^{\frac{3}{4}} \cdot x^1 \cdot x^{\frac{5}{4}} \cdot x^{\frac{6}{4}} \cdot x^{\frac{7}{4}} \cdot \&c.$ Geometrically Proportional; as there are three *Arithmetick Means*, between

$\frac{2}{4}$, and $\frac{6}{4}$; so there are three *Geometrick Means*,

between $x^{\frac{2}{4}}$ and $x^{\frac{6}{4}}$, the Terms in the Geometrick, that are *expounded* by the former Terms,

in the Arithmetick Series. And as $\frac{3}{4}$ is the

first of those Arithmetick Means, so is $x^{\frac{3}{4}}$ the first of the Geometrick Means.

The Reader will easily furnish himself with enough Examples of this kind, or any other, relating to these Matters, after the hints we give him. I chuse all along, to bring Examples in a subserviency to Demonstration
(and

(and therefore as many, and no more than suffice to that end) rather than to heap up a vast number of them, and make them serve instead of Demonstrations, as is too commonly done.

A R T. XVII.

3. The number of Arithmetick or Geometrick Mean Proportionals, that are included between any two Terms, is very easily determin'd. 'Tis certain that the number of Geometrick Means, included between any two Terms, is the same with that of the Arithmetick Means, included between those two Terms, that expound the aforesaid Terms in the Geometrick Series. So that the number of the one being found, that of the other is determin'd also.

Now 'tis plain, that in any Series of Arithmetick continued Proportionals whatsoever, as 0 . 3 . 6 . 9 . 12 . 15 . 18 . &c. the number of *Means* between 0 and any Term, is the same with the number of Places in the whole Series, so far continued as that Term, abating two Places. Thus 18 here is the 7th Place, and there is 5 continued Means between 0 and 18. So also 9 being the 4th place, there can be but two means fall between 0 and 9. Therefore between 9 and 18, there will be but two means, since 9 it self is one of the five Means between 0 and 18. So that the Number of Means between
any

any two Terms whatsoever, is equal to the number of Means between 0 and the remotest Term from 0, abating the number (of Means between 0 and the nearest Term to 0) increas'd by Unity. Thus if 15 be the 6th place from 0 (inclusively) then $6 - 2 = 4$, and there are 4 continued Means between 0 and 15. And taking any other Term as 9, which being the 4th from 0, there are 2 Means between 0 and 9. Now $2 + 1 = 3$, and $4 - 3 = 1$; which, I say, is the number of Means between 9 and 15; or, 'tis all one, if we say that the number of *Means* between any two Terms, is equal to the difference (between the numbers of the places of those Terms) abating Unity. Thus 6 and 12, are the 3d and 5th places; now $5 - 3 = 2$, from whence subtract Unity, and there remains 1 for the number of means between 6 and 12. And so of any other.

These things are general, and hold whatever Terms the Arithmetick Series consists of, whether Integers or Fractions.

In the natural and orderly Progression of numbers from 0, viz. 0. 1. 2. 3. 4. 5. &c. The number of *Means* between 0 and any Term is ever equal to that Term it self, abating Unity; that is, to the next Term below it. Thus 4 is the number of Arithmetick continued Means, between 0 and 5. And 'twould be the same, if 'twere a Fractional Series, whose Denominator

ominator were any number whatsoever, provided the Numerators be the natural Numbers;

as $0. \frac{1}{5}. \frac{2}{5}. \frac{3}{5}. \frac{4}{5}. \frac{5}{5}. \frac{6}{5}. \frac{7}{5}. \&c.$ Thus between

0 and $\frac{7}{5}$, there are six Arithmetick Means.

In these sort of Fractional Series, some of the Terms necessarily come out whole numbers, and whole numbers too in the orderly Progression of $1. 2. 3. \&c.$ And then the number of Means between 0 and $1.$ or 0 and $2.$ or 0 and $3,$ $\&c.$ will be the same as was shewn before, for whole numbers; Or we may express it otherwise, with a particular regard to the Fractional Nature of these Terms, and so say, that it is equal to the Denominator of the Fraction taken once, abating Unity; taken twice, abating Unity; taken thrice, abating Unity; and so

on. *Ex. gr.* $0. \frac{1}{4}. \frac{2}{4}. \frac{3}{4}. \frac{4}{4}. \frac{5}{4}. \frac{6}{4}. \frac{7}{4}. \frac{8}{4}.$

$\frac{9}{4}. \frac{10}{4}. \frac{11}{4}. \frac{12}{4}. \&c.$ Here the Terms

$\frac{4}{4}. \frac{8}{4}. \frac{12}{4}.$ are equivalent to $1. 2. 3;$ and

the common Denominator is $4.$ Now $4 - 1 = 3,$ is the number of Means between 0 and

$\frac{4}{4};$ and $2 \times 4 - 1 = 8 - 1 = 7,$ the num-

ber of Means between 0 and $\frac{8}{4},$ or $2;$ and

$\frac{3}{4} - 1 = 1$, the number of Means between 0 and 3 or $\frac{12}{4}$. And in like manner for any other such Progressions. The number of *Means* between 0 and 1, is ever equal to the Denominator less Unity; and the number of *Means* between 1 and 2, or 2 and 3, or 3 and 4, or any other two Integer Terms (immediately succeeding each other in the order of Numeration) is the same with that between 0 and 1; from whence, the number of Means between 0 and any Integer Term, readily appears to be what was before determin'd, viz. the common Denominator of the Series, taken once, abating Unity; taken twice, abating Unity; taken thrice, abating Unity, &c. And from what has been said, we may easily collect the number of Means between any two Fractional Terms whatsoever, in these sort of Progressions; so that I need not insist any longer upon that Matter.

A R T. XVIII.

4. Whathas been said of Powers when their *Indices* or Exponents, are expressed by Numbers, is as easily applied, when those Exponents are more generally express'd by the Letters of the Alphabet, as *m*, *n*, *p*, and the like. Thus x^m , y^n , express

press those Powers of x and y , which are expounded by m and n , respectively ; and $x^{\frac{1}{m}}$, $y^{\frac{1}{n}}$, when the Exponents are $\frac{1}{m}$, $\frac{1}{n}$, and x^{-m} , y^{-n} , when the Exponents are $-m$, and $-n$. Again x^{-m} , and y^{-n} , are the same with $\frac{1}{x^m}$, and $\frac{1}{y^n}$; and $x^m \times x^n = x^{m+n}$, and $\frac{x^m}{x^n} = x^{m-n}$; and $x^{-m} \times x^{-n} = x^{m-n}$, and $\frac{x^{-m}}{x^{-n}} = x^{-m+n}$; and $x^m \times x^m = x^{2m}$; and x^m rais'd to the Power whose Exponent is n , comesto x^{mn} (that is, 'tis now a Power, whose Exponent is the Product of the two Exponents m and n , multiplied together) and the n Root of the Power x^m being extracted, it comes to $x^{\frac{m}{n}}$ (that is, 'tis now a Power, whose Exponent is equal to the Exponent m divided by the Exponent n . The Reason and Demonstration of all which, is most evident from what has been already done : To bring which the more readily into the Readers Mind, he may consider these (and such like) Series, viz.

	0	m	2m	3m	4m	5m	
Pow.	x	x	x	x	x	x	&c.
Exp.	0.	m.	2m.	3m.	4m.	5m.	&c.

Pow:

$$\text{Pow. } x^0 \cdot x^{\frac{1}{m}} \cdot x^{\frac{2}{m}} \cdot x^{\frac{3}{m}} \cdot x^{\frac{4}{m}} \cdot x^{\frac{5}{m}} \cdot \&c.$$

$$\text{or } x^0 \cdot x^{\frac{1}{m}} \cdot x^{\frac{2}{m}} \cdot x^{\frac{3}{m}} \cdot x^{\frac{4}{m}} \cdot x^{\frac{5}{m}} \cdot \&c.$$

$$\text{Exp. } 0 \cdot \frac{1}{m} \cdot \frac{2}{m} \cdot \frac{3}{m} \cdot \frac{4}{m} \cdot \frac{5}{m} \cdot \&c.$$

$$\text{Pow. } x^0 \cdot x^{\frac{1}{m}} \cdot x^{\frac{2}{m}} \cdot x^{\frac{3}{m}} \cdot x^{\frac{4}{m}} \cdot x^{\frac{5}{m}} \cdot \&c.$$

$$\text{Exp. } 0 \cdot \frac{1}{m} \cdot \frac{2}{m} \cdot \frac{3}{m} \cdot \frac{4}{m} \cdot \frac{5}{m} \cdot \&c.$$

$$\text{Pow. } x^0 \cdot x^{\frac{1}{m}} \cdot x^{\frac{2}{m}} \cdot x^{\frac{3}{m}} \cdot x^{\frac{4}{m}} \cdot x^{\frac{5}{m}} \cdot \&c.$$

$$\text{or } x^0 \cdot x^{\frac{1}{m}} \cdot x^{\frac{2}{m}} \cdot x^{\frac{3}{m}} \cdot x^{\frac{4}{m}} \cdot x^{\frac{5}{m}} \cdot \&c.$$

$$\text{Exp. } 0 \cdot \frac{1}{m} \cdot \frac{2}{m} \cdot \frac{3}{m} \cdot \frac{4}{m} \cdot \frac{5}{m} \cdot \&c.$$

A R T. XIV.

And thus much concerning the Nature, Generation and Properties of *Powers* and their *Exponents*; we may now go on with the Thread of our Discourse in the Doctrine of Fluxions.

Let it be requir'd to find the Fluxion of x^m ; that is, of the Power of x , whose Exponent is

m , suppose $x = v$; and take the small Quan-

tity o , as before; and let ox , and ov be the Increments of the Quantities x and v , generated in the same Moment of time. In the same Mo-

ment that x flows into $x + ox$, the Quantity x be-

comes $x + ox$. And in the same Moment that

G

x

x becomes $x + o \dot{x}$, the Quantity v becomes $\dot{v} + o \ddot{v}$. But $x + o \dot{x}$ is $= x + m x o \dot{x} + \frac{m m - m}{2} x^2 o o \dot{x} \dot{x} + m^3 - 3 m^2 + 2 m$

$\frac{m-3}{6} x^3 o^3 \dot{x}^3$, &c. So that the Augments of the Quantities x and v , generated in the same Moment, are $m x o \dot{x} + \frac{m m - m}{2} x^2 o^2 \dot{x}^2 + m^3 - 3 m^2 + 2 m x o^3 \dot{x}^3$, &c. and

$o \ddot{v}$; and because the increasing Quantities themselves are equal by the supposition; therefore these (*Momentaneous*) Increments of them shall also be equal. Consequently; $m x o \dot{x} + \frac{m m - m}{2} x^2 o^2 \dot{x}^2 + m^3 - 3 m^2 + 2 m$

$\frac{m-3}{6} x^3 o^3 \dot{x}^3$ shall $= \dot{v}$. Therefore making the augmenting Quantity o vanish (in order to have the ultimate Ratio of the Evanescent Increments) we have $m x \dot{x} = \dot{v}$, equal to the Fluxion of the Quantity x^n . Therefore whether the Exponent of the Power be a positive Number or a Negative, and either way a whole Number or a Fraction; that is, whether the Quantity propos'd, be according to the common way of speak

speaking, a Power, or a Root ; the Rule for finding the Fluxion is this, viz. *Subtract Unity from the Exponent of the Power, and set down the Flowing Quantity with the lessen'd Exponent, as a new one ; then multiply this by the Fluxion of the Root, and this Product again, by the whole Exponent of the Power, and the Quantity so produc'd shall be the Fluxion requir'd.* As here I subtract 1 from m the Exponent, and set down the flowing Quantity with the Exponent $m-1$, which then

becomes x^{m-1} ; this I multiply first by x (the Fluxion of x) and this again by m (the whole Exponent) which Product makes $m x^{m-1} x$;

the Fluxion of x^m . In like manner, the Fluxion of any other Powers may be found ; and the Reader may, if he pleases, take this Fluxion found, as a standing Cannon to find others by, only supposing the universal Exponent m to be interpreted by, or equal to some other particular Quantity. Thus if $m = 2$, or 3, or 4, &c. or if $m = \frac{1}{2}$, or $\frac{1}{3}$, or $\frac{1}{4}$, &c. then would $x^m = x^2$, or x^3 , or x^4 , or $x^{\frac{1}{2}}$, or $x^{\frac{1}{3}}$, or $x^{\frac{1}{4}}$, &c. and consequently x^{m-1} would = x^1 , or x^2 , or x^3 , or $x^{-\frac{1}{2}}$, or $x^{-\frac{2}{3}}$, or $x^{-\frac{3}{4}}$, &c.

and therefore $m x^{m-1} x$ would equal $2 x x$, or $3 x x^2$, or $4 x^3$, or $\frac{1}{2} x^{-\frac{1}{2}}$, or $\frac{1}{3} x^{-\frac{2}{3}}$, or $\frac{1}{4} x^{-\frac{3}{4}}$, &c.
 or $\frac{3}{4} x^{\frac{3}{4}}$, or $\frac{1}{2} x^{\frac{1}{2}}$, or $\frac{1}{3} x^{\frac{1}{3}}$, or $\frac{1}{4} x^{\frac{1}{4}}$, &c.
 G 2 or

as is plain, &c. which would be the Fluxions of these Quantities respectively. And as to the Fluxions $\frac{1}{2} x^{-\frac{1}{2}}$, $\frac{1}{3} x^{-\frac{2}{3}}$, $\frac{1}{4} x^{-\frac{3}{4}}$, or any others, where the Exponent of the Flowing Quantity, is less than Unity; 'tis certain that

they may be thus expressed, viz. $\frac{x}{2x^{\frac{3}{2}}}$, $\frac{x}{3x^{\frac{5}{3}}}$,

$\frac{x}{4x^{\frac{7}{4}}}$, for we have demonstrated before, that

the Expressions $x^{-\frac{1}{2}}$, $x^{-\frac{2}{3}}$, $x^{-\frac{3}{4}}$, &c. are equivalent to $\frac{1}{x^{\frac{1}{2}}}$, $\frac{1}{x^{\frac{2}{3}}}$, $\frac{1}{x^{\frac{3}{4}}}$, &c. and universally,

that any Power may be changed from the condition of a Multiplier into the condition of a Divisor, or from that of a Divisor, into that of a Multiplier; if the Exponent of it be changed from Positive to Negative, or from Negative to Positive. Thus, suppose the flowing

Quantity $y^{\frac{n}{m}}$, where the Exponent is $\frac{n}{m}$, and its

Fluxion therefore by the Rule, is $\frac{n}{m} y^{\frac{n}{m}-1} y$, or

because $\frac{n}{m} - 1 = \frac{n-m}{m}$, 'tis $\frac{n}{m} y^{\frac{n-m}{m}}$.

Now if $\frac{n}{m} < 1$, or (which is all one) if $n < m$,

then

then 'tis plain that the Exponent $\frac{n-m}{m}$ is a Negative Quantity ; and therefore, $\frac{m-n}{m}$ shall be a positive Quantity. And, I say, 'tis the same thing, whether the Fluxion be expressed thus $\frac{n}{m} y^{\frac{n-m}{m}} \dot{y}$, where the Quantity y has a Negative Exponent, and is in the condition of a Multiplier (for $y^{\frac{n-m}{m}}$ is multiplied into $\frac{n}{m} \dot{y}$) or whether it be expressed thus $\frac{n \dot{y}}{m y^{\frac{m-n}{m}}}$, where y has a positive Exponent, and is in the condition of a Divisor.

A R T. XX.

If more Examples are thought necessary, the Reader at his leisure may demonstrate from the same Principles that the Fluxion of $x^{\frac{2q}{2q+1}}$ is $\frac{2q}{2q+1} x^{\frac{2q}{2q+1}-1} \dot{x}$; of $y^{\frac{n}{m}}$ is $\frac{n}{m} y^{\frac{n}{m}-1} \dot{y}$; of $y^{\frac{n}{m}}$ (where m the

Exponent of y , has also an Exponent which is n) is $\frac{n}{m} y^{\frac{n}{m}-1} \dot{y}$; which is also the Fluxion of

$\frac{x}{y^m}$ If we have Products, or Quotients of any Powers, of flowing Quantities, we may as easily determine their Fluxions. Thus the Fluxion of xyy , is $2xy\dot{y} + 2y\dot{y}x$; of

$\frac{xx}{yy}$, is $\frac{2xx\dot{y}y - 2y\dot{y}xx}{y^4}$ and universally ;

the Fluxion of $x^m y^n$ is $m x^{m-1} \dot{x} y^n + n x^m y^{n-1} \dot{y}$,

and the Fluxion of $\frac{x^m}{y^n}$, is $\frac{m x^{m-1} \dot{x} y^n - n x^m y^{n-1} \dot{y}}{y^{2n}}$

which general Case for the Readers ease and assistance, I will here add the Demonstration of.

Let $x^m y^n = v$, then $x + o\dot{x} \times y + o\dot{y} = v + o\dot{v}$ (from what has been shewn sufficiently already) That is, $v + o\dot{v} = x^m + m x^{m-1} o\dot{x} + \frac{m m-1}{2} x^{m-2} o\dot{x}^2 \&c.$ multiplied by $y + n y o\dot{y} + \frac{n n-1}{2} y^2 o\dot{y}^2 \&c.$ That is, $x^m y + m x^{m-1} o\dot{x} y + \frac{m m-1}{2} x^{m-2} o\dot{x}^2 y + x^m n y o\dot{y} + \frac{m n n-1}{2} x^m o\dot{y}^2 + m x^{m-1} o\dot{x} n y o\dot{y} + \frac{m m-1}{2} x^{m-2} o\dot{x}^2 n y o\dot{y} +$

$$x \frac{m \dots m}{2} y^{\frac{n-2}{2}} o \dot{y} + m x^{\frac{m-1}{2}} o \dot{x} \frac{m \dots m}{2} y^{\frac{n-2}{2}} o^2 \dot{y} +$$

$$\frac{m \dots m}{2} x^{\frac{m-2}{2}} o \dot{x} \frac{m \dots m}{2} y^{\frac{n-2}{2}} o \dot{y}^2 \&c. = v + o \dot{v}$$

Therefore taking away Equals from both Sides, and dividing the Remainders by the small Quantity o , and lastly making o vanish, (that we may have the last Ratio of the Increments)

we find $m x^{\frac{m-1}{2}} \dot{x} y^{\frac{n-1}{2}} + n y^{\frac{n-1}{2}} \dot{y} x^{\frac{m}{2}} = \dot{v}$, which is therefore the Fluxion sought. Thus also the

Fluxion of $\frac{x^n}{y}$ will be demonstrated, to be what we just now assigned.

But *note*. That the Fluxions of Products or Quotients, of Powers, (supposing that we know how to find the Fluxion of a single Power) may be demonstrated immediately from the general Rules before given, for finding the Fluxions of Products and Fractions. So that in all such Cases as these are, the Reader may serve himself immediately by those Rules, without Recourse to the Original Demonstration; though that Process be so very fine and curious, that it can hardly be repeated too often.

*Some COROLLARIES from
the General Rule for finding the Fluxions of Powers.*

A R T. XXI.

C O R O L L. I.

If any two Powers of Flowing Quantities are equal, those Powers lessen'd by Unity, are in the Ratio compounded of the Exponents, and the Fluxions of the Roots, reciprocally. Let $x^m = y^n$, then by the Rule for the Fluxions of Powers, $m x^{m-1} \dot{x} = n y^{n-1} \dot{y}$, and consequently $x : y :: n y : m x$.

A R T. XXII.

C O R O L L. II.

If the Root of one Power be to the Root of another, as the Exponent of that first Power into the Fluxion of the Root, to the Exponent of the other Power into the Fluxion of its Root; then will those Powers of the Flowing Quantities, be directly proportional to their respective Fluxions. Suppose the Powers x^m and y^n ; and let $x : y :: m x : n y$. Then $n x \dot{y} = m y \dot{x}$, and $n y = m y x$ and

and (multiplying both parts by the Product of some Powers of the Quantities x , y , which Product may be taken at liberty, provided each Quantity has its own proper Exponent, as I

take it here $x^m y^n$, as being the most obvious

Expression) we have $n x^m y^n = m y^n x^{m-1}$, and $m x^m y^n = n x^m y^{n-1}$,

and $n x^m y^n = m y^n x^{m-1}$, from whence, $x : y :: m x^{m-1} : n y^{n-1}$, which two latter Expressions

are the Fluxions of the two former, by the Rule.

Note; If I had multiplied by $x^p y^q$ ex. gr. and expounded p by $m+2$ ex. gr. and q by $n+4$, it would have come to the same Issue; and so of any other. *The Converse of this is also true.*

A R T. XXIII.

C O R O L L. III.

Any two Powers of the same flowing Quantity, divided by the Exponents, alternately taken; are proportional to the Fluxions of those

Powers. *Ex. gr.* $\frac{x^m}{m} : \frac{x^n}{n} :: m x^{m-1} : n x^{n-1}$. Which

is evident, because the Products of the Extreams

and Means, are each $= x^{m+n-1}$.

A R T. XXIV.

C O R O L L. IV.

In a Series of Powers of the same flowing Quantity ; if the Fluents themselves are in continual Geometrical Proportion, the Fluxions shall be a Series compounded of continual Arithmetick, and continual Geometrick Proportionals. Thus if we have this Series of Flu-

ents, viz. x^m x^{m-1} x^{m-2} x^{m-3} x^{m-4} &c. which are Geometrick Proportionals, then we shall have this Series for their Fluxions, viz.

$m x^{m-1}$ $(m-1) x^{m-2}$ $(m-2) x^{m-3}$ $(m-3) x^{m-4}$ &c. Where the Powers of x , are Geom. Proport. and the Quantities multiplied into them, viz. m $m-1$ $m-2$ $m-3$ &c. are Arith. met. Proport. as is most evident.

A R T. XXV.

C O R O L L. V.

In a Series of Powers of the same flowing Quantity ; if the Fluents be a Series compounded of continual Geometrical, and continual Musical Proportionals ; the Fluxions shall be a Series of continual Geometrical Proportionals.

Sup-

Suppose this Series of Fluents, viz.

$$\frac{1}{m} x^m \cdot \frac{1}{m-1} x^{m-1} \cdot \frac{1}{m-2} x^{m-2} \cdot \frac{1}{m-3} x^{m-3} \cdot \text{\&c.}$$

where the Powers of x are in Geom. Proporti-

on and the Quantities $\frac{1}{m} \cdot \frac{1}{m-1} \cdot \frac{1}{m-2} \cdot \text{\&c.}$

are in Musical Proportion; for they are the Reciprocals of $m \cdot m-1 \cdot m-2 \cdot \text{\&c.}$ which are Arithmetically proportional. Now the

Fluxions of these Terms, are $x^{m-1} \cdot x^{m-2} \cdot x^{m-3}$

$x^{m-4} \cdot \text{\&c.}$ as appears by the Rule; and they are evidently in Geometrical Proportion.

N. B. For what Reason the Converse of the first, third, fourth and fifth *Corollaries*, will not hold true Universally, as well as the Converse of *Cor. II.*; we shall see afterwards, when we come to speak of the Inverse Method. For we shall shew there, that we cannot argue so unlimitedly, from the Fluxions back to the Fluents, as we can from the Fluents themselves to the Fluxions: And consequently, a Reasoning that may be true in the latter Case, may be false in the former.

A R T.

A R T. XXVI.

C O R O L L. VI.

The Fluxion of any Power (the Exponent of which is positive) of a flowing Quantity, is the second Term of an infinite Series, made by raising a Binomial to the same Power; which Binomial, is the Root of the same Power + the Fluxion of the Root.

If the Power x^m were proposed, then $m x^{m-1}$ will be its Fluxion. Now if we raise $x + \dot{x}$ to the Power m ; then will $x + \dot{x} = x^m + m x^{m-1} \dot{x} + \frac{m(m-1)}{2} x^{m-2} \dot{x}^2$ &c. the 2d Term of which Se-

ries, is the Fluxion of x^m , as before found. If the Exponent were Negative, that is, if it had been x^{-m} ; yet the second Term of the Series $x + \dot{x} = x^{-m} - m x^{-m-1} \dot{x} + \frac{m(m+1)}{2} x^{-m-2} \dot{x}^2$ &c.

will be the Fluxion of x^m . But then in this Case x (the first Member of the Binomial) is not the Root of the Power x^m ; as in the former Case, it was of the Power x ; and so the Binomial here, is not the the Root of the Power

Power + the Fluxion of that Root, as 'twas there.

N. B. In such Series as these, which (if the Exponent m be a Fraction) run on infinitely; we have a view of so many several Species or Orders of *Infinity*; that is, the Terms of these Series, are still infinitely greater one than the other; or the foregoing Term is infinitely great in respect of that which follows. This is very obvious. For taking the Series rais'd

$$\text{from the Binomial } x + \dot{x}^{\frac{m}{m-1}}, \text{ which is } = x^{\frac{m}{m-1}} + m x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}} + \frac{m \times m-1}{2} x^{\frac{m-2}{m-1}} \dot{x}^2 + \frac{m \times m-1 \times m-2}{3} x^{\frac{m-3}{m-1}} \dot{x}^3 + \&c.$$

'tis evident if we do not consider the determinate Quantities (or Coefficients) m , $m \times m-1$, &c. the other Fluxional Expressions

are in continued Geometrical Proportion, viz.

$$x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}} x^{\frac{m-3}{m-1}} \dot{x}^2 x^{\frac{m-4}{m-1}} \dot{x}^3 x^{\frac{m-5}{m-1}} \dot{x}^4 \&c. \text{ —ls. But}$$

$x^{\frac{m}{m-1}}$ is infinite in respect of $x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}}$; therefore

$x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}}$ is infinite in respect of $x^{\frac{m-2}{m-1}} \dot{x}^2$, and

$x^{\frac{m-2}{m-1}} \dot{x}^2$ in respect of $x^{\frac{m-3}{m-1}} \dot{x}^3$, and $x^{\frac{m-3}{m-1}} \dot{x}^3$ in respect

of $x^{\frac{m-4}{m-1}} \dot{x}^4$, But if $x^{\frac{m}{m-1}}$ be infinite in respect of

$x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}}$, it shall also be infinite in respect of $m x^{\frac{m-1}{m-1}} \dot{x}^{\frac{m-2}{m-1}}$

$m x^{\frac{m-1}{2}}$, and this Infinite in respect of m ;
 $\frac{m-1}{2} x^{\frac{m-2}{2}}$, and this Infinite in respect of m ;
 $\frac{m-1}{2} \times \frac{m-2}{3} x^{\frac{m-3}{2}}$, &c. So that this whole
 Series presents us with an infinite number
 of Terms; in which the foregoing is still in-
 finitely great, in respect of that which follows.
 If the Exponent m were a whole number, the
 Terms are all Infinite in respect of one another, as
 before; only the Series will stop at a certain
 number of Terms, and not run on infinitely as
 in the other Case.

A R T. XXVII.

The Fluxions of compound Quantities;
 are all easily deduced from the same general
 Rule; whether those compound Quantities are
Surd, or any Powers of *Rational* ones. Thus the

Fluxions of $a y - x x$ (that is, of the third
 Power or Cube of the Quantity $a y - x x$) is
 $= 3 a y - x x \times a y - 2 x x$, that is, 'tis equal
 to the Quantity it self (rais'd to a Power
 whose Index is less by Unity, than before)
 multiplied into the Fluxion of the Root. The

Fluxion of $\sqrt{x y + y y}$, or $x y + y y$, is

$$\frac{xy + yy}{2xy + yy} = \frac{yx + xy + 2yy}{2xy + yy}$$

The Fluxion of $a^n + x^m$, or $\sqrt[n]{a^n + x^m}$, is =

$$\frac{1}{q} \frac{1}{a^n + x^m} \frac{1}{q} \frac{m-1}{m} x^{m-1} = \frac{m-1}{m} \frac{x^{m-1}}{q a^n + x^m \frac{q-1}{q}}$$

The

Reasons of all which, both as to the Operation and Notation (as also for any other Examples that can be propos'd) may be presently deduc'd from what has been hitherto demonstrated. Likewise, from the same Principles, 'twill be as easie to find the Fluxions of Quantities compos'd of *Rational* and *Surd* Quantities; so that I need not dwell any longer upon this Subject. I'll only add, that in such cases as this last mention'd, as also in others, where the Quantities we have to deal with, are very large; and much compounded; to facilitate the Operation, 'twill be convenient, to substitute some other Quantities instead of those large ones, and so work with them, and at the end of the Operation, restore the others into their respective places again. Thus suppose 'twere requir'd to find

the Fluxion of this Quantity $bxx + cax + eaa$
 $\sqrt{bxx + cax + eaa}$. Put this Quantity = z , the *Ra-*
tional part $bxx + cax + eaa = v$, and the
Surd

Surd Part $\sqrt{xx + aa} = y$. Then will $vy = z$, and $v\dot{y} + y\dot{v} = \dot{z}$; and then in the Places of $v, y, \dot{v}$, and $\dot{y}$ substitute their respective Values, and the Equation so arising will $\dot{z} =$ or the Fluxion of the compound Quantity propos'd.

A R T. XXVIII.

Before I proceed farther, I would observe, that in all the preceding Operations about the Fluxions of Flowing Quantities, we have suppos'd the several flowing Quantities, to be all of them *Increasing*, or becoming still bigger and bigger. But it may happen so, that while some of them are *Increasing*, others at the same time may be *Decreasing*, or at least may be considered as doing so; and in that Case, I say, that the Fluxions of those Quantities are to be look'd upon as *Negative*, in respect of the Fluxions of those other Quantities, that *Increase*. Thus, in the Product xyz , if we consider all the Flowing Quantities x, y, z , as *Increasing*, the Fluxion will be (as we have shewn) $x\dot{y}z + x\dot{y}z +$

$x\dot{y}z$. But if we suppose x to *increase*, while y and z *decrease*: Then, I say, that the Terms in which the the Fluxions of y and z are found, must have the contrary Signs, to what they have when we suppose y and z to increase as well

as x . And therefore, in this latter case, the Fluxion will be $x y z - x y z - x y z$; and the like in other cases of the same Nature: The reason of which is this. In order to find the

Fluxion we substitute for x increasing $x + o x$, or x with its positive Momentum; but because, by the Hypothesis, the Quantities y and z are decreasing at the same time; therefore we must

substitute for y and z the Expressions $y - o y$ and $z - o z$; that is, y and z , with their Momenta Negative; by which Substitutions, in the Equation resulting, the Terms in which the Fluxions of y and z are found will appear with their Signs thus chang'd.

SECTION V.

Of. Second Fluxions.

AND now (I think) we have demonstrated all the Fundamental Points in the Doctrine of Fluxions, and deduc'd the necessary Rules for Practice, from those Demonstrations. We may from hence go on with ease to the second, third, and any other Fluxions; neither are there any new Difficulties to be met with, saving those of the meer Operation; nor any other Rules to be used but those we have already laid down. 'Tis only a Repetition of the first Operation, as Mr. Newton tells us in the fore-mention'd *Treat.* But it may be worth while to demonstrate this, and shew, that the Rules here will come out the very same as before.

A R T. I.

In general : Suppose this Equation were given, viz. $zy - xx + ax = 0$, and 'twere requir'd to find the Equation determining the relation of the second Fluxions. Putting

$0z,$

$\dot{o}z, \dot{o}y, \dot{o}x$, the Increments of the flowing Quantities z, y, x , generated in the same Moment ; the first Operation brings it to $zy + yz - 2xx + ax = 0$, which is the Equation that gives the Relation of the first Fluxions. Now the Fluxion of this Equation $zy + yz - 2xx + ax$, will be that which will determine the Relation of the second Fluxions, as is most manifest.

Here then we are to consider $z, y, x, \dot{z}, \dot{y}, \dot{x}$, as so many several flowing Quantities, whose

Momentaneous Increments are $\dot{o}z, \dot{o}y, \dot{o}x$, $\ddot{o}z, \ddot{o}y, \ddot{o}x$, respectively. Also in the Equation

$zy + yz - 2xx + ax = 0$, putting $z + \dot{o}z$, $y + \dot{o}y$, $x + \dot{o}x$, $\dot{z} + \ddot{o}z$, $\dot{y} + \ddot{o}y$, $\dot{x} + \ddot{o}x$ (or the Powers of these) in the room of $z, y, x, \dot{z}, \dot{y}, \dot{x}$, (or their Powers, respectively) we shall have this Equation, $z\dot{y} + \dot{o}y\dot{z} + \dot{o}z\dot{y} + \ddot{o}o\dot{z}\dot{y} + y\dot{z} + \dot{o}z\dot{y} + \ddot{o}o\dot{y}\dot{z} - 2x\dot{x} - 2\dot{o}x\dot{x} - 2\ddot{o}o\dot{x}\dot{x} + a\dot{x} + \ddot{o}o\dot{x} =$ nothing. From whence taking away $z\dot{y} + y\dot{z} - 2x\dot{x} + a\dot{x} =$ nothing, and dividing all the remaining Terms by the little Quantity $\dot{o}$, we have $\dot{z}\dot{y} + \dot{z}\dot{y} + \ddot{o}z\dot{y} + \dot{z}\dot{y} + y\dot{z} + \ddot{o}y\dot{z}$

$-2\dot{x}\ddot{x} - 2x\ddot{x} - 2o\ddot{x}\ddot{x} + a\ddot{x} = \text{nothing}$.
 Lastly, making the little Quantity o vanish,
 we have then (striking out all the Terms into
 which that is multiplied) $z\ddot{y} + 2\dot{y}\dot{z} + y\ddot{z} -$
 $2\dot{x}\ddot{x} - 2x\ddot{x} + a\ddot{x} = \text{nothing}$, the Equation
 desir'd which gives the Relation of the second
 Fluxions.

SCHOL. I.

We express the *Momenta* here when we have
 to do with second Fluxions, after the same
 manner as we did for first Fluxions. That is
 as $o\dot{z}, o\dot{y}, o\dot{x}$, are supposed the Increments of
 z, y, x , which in the same Moment of time
 therefore, become $z + o\dot{z}, y + o\dot{y}, x + o\dot{x}$;
 so also $o\ddot{z}, o\ddot{y}, o\ddot{x}$, are the *Momenta* or Incre-
 ments of the Fluents z, y, x , which conse-
 quently in the same Moment, are augmented
 into $z + o\ddot{z}, y + o\ddot{y}, x + o\ddot{x}$. And after the
 same manner, is the Expression to be suited to
 third, fourth, or any other order of Fluxions,
 observing only the Peculiarities of the Nota-
 tion in each kind.

A R T.

A R T. II.

From the Operation before demonstrated, the Reader may easily find the way of proceeding in any other case propos'd ; and determine the Fluxions of any Fluxionary *Products, Fractions* or *Powers* whatsoever. *Ex. gr.* Let the Product zy be propos'd. Put $zy = v$, then zy

$+ ozy + oyz + ooyz = v + ov$, consequently

(because $zy = v$) $ozy + oyz + ooyz = ov$

and $zy + yz + oyz = v$, and making o vanish,

$zy + yz = v =$ the Fluxion of zy , and consequently equal to the second Fluxion of that

Quantity whose first Fluxion is zy . Again,

suppose the Fraction $\frac{yy}{x}$ were propos'd. Put $\frac{yy}{x}$

$= v$, then $yy = vx$, and $yy + oyy + oyy +$

$ooyy = vx + ovx + ovx + oovx$, and taking away equals, and dividing by o , and

making o vanish, we shall find $yy + yy = vx$

$+ vx$, and putting in the value of v ; which is

$\frac{yy}{x}$, it will be $yy + yy = \frac{yy}{x}x + \frac{yy}{x}x$, and lastly

ly reducing all, we shall have $\frac{\dot{x} \ddot{y}^2 + \dot{x} \ddot{y} \dot{y} - \dot{y} \dot{y} \ddot{x}}{\dot{x}^2}$

$= \dot{v}$; and therefore equal, to the Fluxion of the Fluxionary Fraction $\frac{\dot{y}}{\dot{x}}$. Thus also the

Fluxion of this Fraction $\frac{\ddot{y}}{\dot{x}}$, will be found by

the same way of reasoning to be $\frac{2 \ddot{y} \dot{y} \dot{x} - \dot{y}^2 \ddot{x}}{\dot{x}^2}$;

and that of $\frac{\dot{y}}{\dot{x}}$, will be $\frac{\ddot{y} \dot{x} - \dot{x} \ddot{y}}{\dot{x}^2}$; and that

of $\frac{\dot{x}}{\dot{y}}$, or $\frac{1}{\dot{y}}$, will be $-\frac{\ddot{x}}{\dot{x}^2}$, or $-\frac{\ddot{y}}{\dot{y}^2}$; and

the like in other cases.

Thus also in Fluxionary Powers. The Fluxion

of $\dot{x}^2$ (or the Square of the Fluxion of x) is $2\dot{x}\ddot{x}$,

of $\dot{x}^3$ or $\dot{x}^4$, is $3\dot{x}^2\ddot{x}$ or $4\dot{x}^3\ddot{x}$; and universally

of $\dot{y}^m$, is $m\dot{y}^{m-1}\ddot{y}$. The Fluxion of $\dot{x}^{\frac{1}{2}}$, is

$\frac{1}{2}\dot{x}^{-\frac{1}{2}}\ddot{x} = \frac{\ddot{x}}{2\dot{x}^{\frac{1}{2}}}$, of $\dot{x}^{\frac{1}{4}}$, or of $\dot{x}^{-\frac{2}{3}}$, is

$\frac{1}{4}\dot{x}^{-\frac{3}{4}}\ddot{x} = \frac{\ddot{x}}{4\dot{x}^{\frac{3}{4}}}$, or $-\frac{2}{3}\dot{x}^{-\frac{5}{3}}\ddot{x} = -\frac{2}{3}\frac{\ddot{x}}{\dot{x}^{\frac{5}{3}}}$ and

universally, of $\dot{x}^{\frac{1}{m}}$ is $\frac{1}{m}\dot{x}^{\frac{1}{m}-1}\ddot{x}$. The Fluxion

of $\sqrt{\dot{x}^2 + \dot{y}^2}$, that is of $\dot{z}^2 + \dot{y}^2$, is equal to

$$\frac{2\ddot{z} \ddot{x} + 2\ddot{y} \ddot{y} \times \frac{2}{2} \sqrt{\ddot{z} + \ddot{y}}}{\sqrt{\ddot{z} + \ddot{y}}} = \frac{\ddot{z} \ddot{z} + \ddot{y} \ddot{y}}{\sqrt{\ddot{z} + \ddot{y}}}$$

the Demonstration of which, as of the rest of the Examples, is too obvious to need any insisting on't. I set these things down purely to shew the Reader, that he has no need of any other Rules but those already deduc'd, to work by, when the Expressions (whose Fluxions are to be determin'd) appear in the form of Fluxions, themselves. For I don't know but some who would boldly enough venture upon such an

Expression, *Ex. gr.* as y^n ; might boggle at the appearance of the Power of a Fluxion, and not know what to do with such a one as that

y^n . Upon which account, I thought some few parallel Cases in this Nature, might (not un-
usefully, at least) be added.

S C H O L. II.

It may possibly be conceiv'd that such Examples as those already produc'd, are not so genuine and pertinent to the matter of second Fluxions, as might be requir'd. For here indeed we find the Fluxion of a Fluxionary Expression; but then the Question may return; *What is this the second Fluxion of?* 'Tis true, that

that $3x^2$, *Ex. gr.* is the Fluxion of the Fluxionary Expression x^3 , but if we pretend to call it a second Fluxion, should there not be some first flowing Quantity assigned, of which this

Term $3x^2$, is the second Fluxion.

To this, I think, there needs no more be said, than thus much; that to find the Fluxion of an Expression, that is it self a Fluxion; is by consequence to find the second Fluxion, of that flowing Quantity (whatever it be) whose first Fluxion, is the the Expression propos'd. 'Tis no matter what that flowing Quantity is, or whether it be expressly given or assign'd, or no; 'tis sufficient to conceive that the Fluxionary Expression given, is the first Fluxion of some flowing Quantity or other; for then the Fluxion of this Fluxionary Expression, is the second Fluxion of that flowing Quantity; or if the Fluxionary Expression were a second, third, or fourth, &c. Fluxion; then its Fluxion wou'd be the third, fourth, or fifth, &c. Fluxion, of that flowing Quantity thus understood. For if that flowing Quantity it self were really assign'd, the second, third, fourth, &c. Fluxions of it, would be no other, than what by these Operations we determine.

A R T. III.

If therefore we will proceed strictly in this matter, according to the import of Words and Names; the flowing Quantities themselves must be previously assigned, and so by repeated Operations, their first, second, third, &c. Fluxions found. *Ex. gr.* suppose we were to find the second Fluxion of xx . Let $xx = v$; then $xx + 20x\dot{x} + 00\dot{x}\dot{x} = v + 0\dot{v}$, and so $2x\dot{x} = \dot{v}$, for the first Operation; that is, for the first Fluxion of xx . Again, taking this Equation $2x\dot{x} = \dot{v}$, and working as before, we shall have (substituting $2x + 20\dot{x}$ for $2x$, and $\dot{x} + 0\ddot{x}$ for $\dot{x}$, and $\dot{v} + 0\ddot{v}$ for $\dot{v}$) $2xx + 20xx\dot{x} + 20xx\ddot{x} + 200xx\ddot{x} = \dot{v} + 0\ddot{v}$, and consequently $2xx + 2xx\ddot{x} = \ddot{v}$, and therefore equal to the second Fluxion of xx . If the Operation were to be repeated and we were to find the third Fluxion of xx ; that is, the second Fluxion of $2xx$, or the first Fluxion of $2x^2 + 2xx$. Then in this last Equation, viz. $2xx + 2xx\ddot{x} = \ddot{v}$, instead of x , $\dot{x}$, $\ddot{x}$, and $\ddot{v}$, putting $x + 0\dot{x}$, $\dot{x} + 0\ddot{x}$, $\ddot{x} + 0\ddot{\ddot{x}}$, and $\ddot{v} + 0\ddot{\ddot{v}}$ respectively, we shall find (by the same

Ope

Operations hitherto used) $6xx + 2xx = v =$
the third Fluxion of xx . Thus also if the Quan-
tity z , were propos'd; then at the first Opera-
tion we should have $4z^3 = v$; at the second;
 $12zz + 4z^3 = v$; and at the third, $24zz +$
 $36z^2 + 4z^3 = v$; and so by repeated Ope-
rations we may carry it on as far as we please.
Neither is there any thing else to be observed,
but only the making due and just Substitutions
of the several flowing Quantities with their re-
spective Momenta or Increments, in the room
of those flowing Quantities themselves, in the
Equation we work with. For then all the rest,
is the meer work of Reduction, and throwing
out Evanescent Terms; the way and reason of
which has been sufficiently shewn. But then,
the demonstrative way of going to work being
thus explain'd, the Reader will have no occa-
sion to proceed, in every Example he takes in
hand) by all these foremention'd Steps; but
may do it after a much shorter and quicker
manner, by having his Eye upon the Rules for
first Fluxions. For (as I said before) they
are general, and hold in all Orders and Kinds
of Fluxions imaginable.

A R T.

A R T. IV.

And 'twould be the same if (instead of descending below our vulgar flowing Quantities, to Fluxions) we went above them, to those Superior Orders of Quantities, which our ordinary Quantities are the Fluxions of. Thus according to the Notation by Strokes or Dashes, over the Letters, which was before described ;

suppose the Quantity z^m were propos'd ; the Fluxion of it is mz^{m-1} ; of $z^{\frac{n}{m}}$, is $\frac{n}{m} z^{\frac{n}{m}-1}$;

of z^m , is mz^{m-1} ; of $z^{\frac{n}{m}}$, is $\frac{n}{m} z^{\frac{n}{m}-1}$; of x , is x ;

of $\frac{z}{y}$, is $\frac{yz - y^2 \frac{z}{y}}{y^2}$; for the Fluxion of the

Numerator z is z , and the Fluxion of the Denominator y is y , and consequently according

to the Rule, the Fluxion of the Fraction $\frac{z}{y}$

is the Expression before determined. And thus the Reader may easily proceed in any Examples of Computation, according to this Notation also.

I shall now subjoin two or three Rules or Observations, that may be of some use in the management of second, third, &c. Fluxions, and so conclude this Section.

A R T. V.

i. When we proceed to second, third, and the following Fluxions, it may be convenient to suppose some one of the flowing Quantities, to *flow uniformly*; and consequently to substitute 1 for the first Fluxion of it, and 0 for the second, and all the other Fluxions (of it) that may follow. By *flowing uniformly*, I mean; flowing so, as that the Increments gotten or produced, in equal Particles of time, may be equal; or that the flowing Quantity, in equal Particles of time, be equally augmented. In this supposition of an equable uniform Flux, the Increments generated in equal Particles of time, being equal to one another, will therefore be represented naturally by a *standing Quantity*; and consequently for any such Increment we may substitute *Unity*; and the Fluxion being (*Proxime*) as the Increment, it follows, that we may substitute *Unity* for that Fluxion. But further; because the Fluxion of a standing Quantity is = to nothing, and the Increment of an uniformly flowing Quantity, is a standing Quantity, to which Increment the Fluxion is proportional; there-

Therefore the Fluxion of that Fluxion will be equal to nothing; and consequently we may substitute 0 in the room of the second, and all the succeeding Fluxions of an uniformly flowing Quantity. By this means, these Fluxionary Equations are more concisely expressed; some of the Terms being shortned, and others quite vanishing. Thus in the Equation zy^3

$$-z^4 + a^4 = 0; \text{ for the Equation determining the relation of the third Fluxions; that is, at the third Operation we have } z^{\cdot\cdot\cdot 3}y^{\cdot\cdot\cdot 2} + 9zy^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 18zy^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 3zy^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 18zy^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 6zy^{\cdot\cdot\cdot 3} - 4zz^{\cdot\cdot\cdot 3} - 36zz^{\cdot\cdot\cdot 2} - 24z^{\cdot\cdot\cdot 3}z = 0.$$

Now if we suppose the Quantity z *ex. gr.* to flow uniformly, and consequently $z = 1$, and $\dot{z}$, and $\ddot{z} = 0$; then four Terms go out entirely, and the Equation is reduced to this, *viz.* $9yy^{\cdot\cdot\cdot 2} + 18y^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 3zy^{\cdot\cdot\cdot 2} + 18zy^{\cdot\cdot\cdot 2}y^{\cdot\cdot\cdot 2} + 6zy^{\cdot\cdot\cdot 3} - 24z = 0.$

A R T. VII.

In all such Equations as this, just now written, we are, notwithstanding, to conceive the Fluxions in the several Terms, to be all of the same

same Rank and Order ; that is, to be all of the first, second, and third, &c. Order, and not one of the first Order, and another of the second, or another, perhaps, of the third, or any other degree. And where it is actually otherwise in the Equation, we are to fill up and compleat those Orders that are defective, by the Fluxions (of a Quantity uniformly flowing) imagin'd to be multiplied into them. Terms of the *first Order*, are first Fluxions, as

x, y, z , or any Quantities into which these are multiplied ; of the second Order, are either second Fluxions, or Squares, or Rectangles of

first Fluxions, as y, y^2, yz , or any Quantities into which these are multiplied : Of the third Order are either third Fluxions, third Powers of first Fluxions, Rectangles of second Fluxions multiplied into first, or Solids of first

Fluxions, as y, z, yy, yz , or any Quantities multiplied into them, and so in proportion as to the succeeding Orders. Now, I say, when it happens that the Terms in an Equation, are not all of the same Order ; we must suppose what is wanting to be supplied by the Fluxion (or some Powers of the Fluxion) of a Quantity flowing uniformly. Thus in the Equation, at the end of the foregoing Paragraph ; the

last

last Term of all, viz. $24z$ is in none of the (enumerated) Orders of Eluxionary Terms, whereas the two first Terms are of the second Order, and the three middle ones are of the third. To make all alike and Homogeneal therefore, we'll suppose z to flow uniformly, and so mul-

tiply z into the last Term, and z into the two first, which brings them all up to the third Order, and then they stand thus $9zyy + 18zyy + 3zyy + 18zyy + 6zy + 24zz = 0$.

Farther, when an Equation is brought into second, &c. Fluxions, 'tis left to discretion, to chuse what Quantity we will suppose to flow uniformly. But then this is to be observ'd, that more Terms may vanish out of the Equation, by supposing one Quantity to flow uniformly, than by supposing another to do so: And, therefore, if we respect that end of contracting the Equation, we must make that Quantity to flow uniformly, which will strike the most Terms out; that is, that Quantity whose second (or any inferiour) Fluxion, occurs oftener, than any such Fluxion of another

Quantity. *Ex. gr.* In the Equation $zy + 9zyy + 9zyy + 18zyy + 3zyy + 18zyy + 6zy - 4zz - 36zzz - 24zz = 0$,
if

if we put z to flow uniformly, there will four Terms go out, whereas if we suppose y to do so, we shall have but three Terms out of the Equation; the Reason of which (if 'twere worth while) were very easie to be shown; and any one may deduce it from the consideration of the Repetitions of the flowing Quantities z and y in the Original Equation.

A R T, VII,

3. If a Quantity flows uniformly, its *first* Power will have a standing first Fluxion, and no second Fluxion; its second Power, a standing second Fluxion, and no third Fluxion; its third Power, a standing third Fluxion, and no fourth; its fourth Power, a standing fourth Fluxion, and no fifth; and so on *ad Infinitum*; the order of the *standing Fluxion*, will be of the same denomination, with the order of the *Power* of the uniformly flowing Quantity. Suppose z to flow uniformly, then will

$$\begin{aligned} z &= v; \text{ then } n z^{\overline{n-1}} = v, \text{ and } n z^{\overline{n-1}} = z + m \overline{n-1} \\ z^{\overline{n-2} \cdot 2} &= v, \text{ and } n z^{\overline{n-1}} = z + n^2 \overline{n-2} \cdot 2 \\ + 2m \overline{n-2} \cdot 2 &= z z + n \overline{n-2} \cdot 2 \times m \overline{n-3} \cdot 3 \\ &= v \end{aligned}$$

$= \dot{v}$; and by repeated Operations we may carry it as far as we please. Now if $n=1$, then the value of $\dot{v}$ or the first Fluxion, will $= 1$; also the value of $\ddot{v}$ the second Fluxion, will $= 0$, for $\ddot{z} = 0$, and $nn - n = 1 \times 1 - 1 = 1 - 1 = 0$; also $\ddot{v}$ will $= 0$, for $\ddot{z}$, and $\ddot{z} = 0$, and $n - 2 \times nn - n = 1 - 2 \times 1 \times 1 - 1 = -1 \times 1 - 1 = -1 \times 0 = 0$.

If $n=2$, then $\dot{v} = 2z\dot{z}$, a flowing Quantity, and $\ddot{v} = 2\ddot{z}$, a standing Quantity, and $\ddot{v} = 0$, because $n - 2 \times nn - n = 2 - 2 \times 4 - 2 = 0$. If $n=3$, then $\dot{v} = 3z^2\dot{z}$, a flowing Quantity, and $\ddot{v} = 6z\ddot{z}$, a flowing Quantity, and $\ddot{v} = 6\ddot{z}$, a standing Quantity; and therefore $\ddot{v}$ would $= 0$. From whence, I believe, what was asserted, may be pretty clear.

A R T. VIII.

4. And from hence now, it will follow, that taking any flowing Quantity as z^n , if we suppose z to flow uniformly, that then the first
1 second,

second, third, &c. Fluxions of the Term z^n , shall be as the second, third, fourth, &c. respective Terms of an infinite converging Series, form'd by raising the Quantity $z + a$ small

Quantity o , to the same Power n . For $z + o$

$$= z^n + n z^{n-1} o + \frac{n(n-1)}{2} z^{n-2} o^2 + \dots$$

$$\frac{n^3 - 3nn + 2n}{6} z^3, \text{ \&c. Now if } z$$

flow uniformly, we have shewn in the foregoing Paragraph that its first, second, third, &c.

Fluxions, are expressed by the Terms $n z^{n-1}$, $n(n-1) z^{n-2}$, $n(n-1)(n-2) z^{n-3}$, &c.

$$n z^{n-1}, n(n-1) z^{n-2}, n(n-1)(n-2) z^{n-3}, \text{ \&c. (or because } z = 1, \text{ and so } z^2, z^3, \text{ \&c.} = 1)$$

by the Terms $n z^{n-1}$, $n(n-1) z^{n-2}$, $n(n-1)(n-2) z^{n-3}$, &c.

But also because o is a standing Quantity, and the Powers of z are the same, in the respective Terms on both sides; therefore they are ever (comparing any two correspondent ones together) in a certain standing

Ratio, viz. $n z^{n-1} : n(n-1) z^{n-2} :: o : 1$, and

$$\frac{n(n-1)}{2} z^{n-2} : \frac{n(n-1)(n-2)}{6} z^{n-3} :: \frac{1}{2} o o : 1. \text{ and}$$

&c.

Q. E. D.

$$\frac{n^3 - 3n^2 + 2n}{6} : \frac{n^3 - 3n^2 + 2n}{6} :: \frac{n^3 - 3n^2 + 2n}{6} : \frac{n^3 - 3n^2 + 2n}{6}$$

And so of the rest, if the Series were continued, so that these Terms are respectively proportional to one another.

SECTION VI.

Of the Fluxions of Logarithms and Exponential Quantities.

A R T. I.

THOUGH we have already treated something largely of the Operations, in finding the Fluxions of flowing Quantities, yet there remains something more, necessary to be said upon that Head, and which the young Geometer ought to be acquainted with. And that is, the way of finding the Fluxion of such Powers, whose Exponents are themselves variable Quantities. Now these will require quite other Rules and Methods, for the handling of them, than those that we have hitherto been concern'd withal, where the Exponents were al-

ways supposed to be constant Quantities. For, the Exponents being here flowing Quantities, as well as the Quantities, whose Exponents they are, we may imagine that the Rules that easily serve in the one case, will never be able to reach the other.

L E M M A.

The Fluxion of any Logarithm is equal to the Fluxion of the absolute Number or Quantity (whose Log. it is) divided by that same Number.

Let the Quantity $x + 1$ be propos'd; the Fluxion of its Logarithm shall be $\frac{\dot{x}}{x+1}$; for the Fluxion of the Quantity it self is $\dot{x}$, and this Fluxion divided by the Quantity, gives $\frac{\dot{x}}{x+1}$. The truth of which is thus made out.

The Log. of $1+x$, is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$, &c.

And so the Fluxion of it, is $\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x}$, &c. But $\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x}$

&c. $= x \times 1 - x + x^2 - x^3$, &c. which is $= \frac{\dot{x}}{x+1}$.

Note

Note, the Logarithms of any Quantities, are, for Brevity's sake, exprest by the Letter *l* (with 2 Points after it) fix'd before a Quantity, and the Fluxions of those Logs. by the same Letter *l* with two Points, one above, and the other under a Line drawn over Head. Thus

$l: \overset{\text{---}m}{x+a}$ denotes the Log. of $\overset{\text{---}m}{x+a}$, and

$l: \overset{\cdot}{\underset{\cdot}{x+a}} \overset{\text{---}m}{}$ the Fluxion of that Log. Also $l: \overset{\cdot}{\underset{\cdot}{x+a}}$

denotes the Log. of the Quantity x ad-

ded to the m Power of a , and $l: \overset{\cdot}{\underset{\cdot}{x+a}} \overset{\text{---}m}{}$ the Fluxion of the Log. of the same; the Line that is drawn between the two Tittles or Points, being shorter in this last Example, where the Exponent m affects but one of the two Quantities, then in the foregoing one, where it affects both, and where the Line consequently is drawn from the one to the other.

A R T. II.

From hence now the Fluxions of the Logs. of all sorts of Powers, and the Fluxions of any Powers of Logs. may with ease be found.

$$\text{Ex. gr. } l: \overset{\cdot}{\underset{\cdot}{xx+yy}} = \frac{2x\dot{x} + 2y\dot{y}}{xx+yy}, \text{ and } l: \overset{\cdot}{\underset{\cdot}{ax^2+x^3}} = \frac{2ax\dot{x} + 3x^2\dot{x}}{ax^2+x^3} = \frac{2a\dot{x} + 3x\dot{x}}{ax+x^2};$$

I 3 and

and the like in other Cases, a multitude of which need not be heaped up. This one general Example may serve instead of a great many. Let it be required to find the Fluxion of the Log. of this Quantity $z^n - x^n$. I say,

that $l : z^n - x^n$ is $= \frac{p}{mz} \dot{z} + \frac{n-1}{pnx} \dot{x}$. For

the Fluxion of the Quantity (by the Rules of direct Method) $z^n - x^n$ is $= \frac{p}{mz} \dot{z} - \frac{n-1}{pnx} \dot{x}$, and this divided by $z^n - x^n$, comes to what was just now determin'd.

A R T. III.

The Powers of the Logs. of any Quantities

are thus expressed, viz. $l : x$, $l : a + x$, that is, the Log. of x , rais'd to the Power whose Exponent is n , and the Log. of $a + x$, rais'd to the same Power; and so of others. And the Fluxions of these Powers of Logs, may be

represented thus, viz. $l : a + x$, and $l : a + x^n$, the former of which denotes the Fluxion of the n Power of the Log. of $a + x$; and the latter, the Fluxion of the m Power of the Log. of

of $a+x$. Which Notation the Reader will do well to observe carefully, to avoid confounding himself, which otherwise he may easily do. Now to find the Fluxions of the Powers of Logs. is no more than to proceed according to the general Rule, for the Fluxions of the Powers of any flowing Quantities, which we have deliver'd and demonstrated before.

Ex. gr. Suppose we were to find the Fluxion of $l: \frac{m}{a+x}$; I say, that $l: \frac{m}{a+x} = m l: \frac{m-1}{a+x} \times \frac{x}{a+x}$. For the flowing Quantity here to be consider'd, is $l: \frac{m}{a+x}$, or the m Power of the Log. of $a+x$. According to the forementioned Rule therefore, we must bring the flowing Quantity to a Power less by Unity, which is $l: \frac{m-1}{a+x}$, then multiply it into the Exponent of the Power, viz. m , which makes $m l: \frac{m-1}{a+x}$; lastly, the Root of the Power $l: \frac{m}{a+x}$, is $l: \frac{1}{a+x}$, and therefore taking the Fluxion of this Root, which (from the Lemma) is $\frac{x}{a+x}$, and multiplying it into $l: \frac{m-1}{a+x}$, we have $m l: \frac{m-1}{a+x} \times \frac{x}{a+x} = l: \frac{m}{a+x}$, or the Fluxion of the

m Power of the Log. of $a+x$. Thus also

$l: a+x$, that is the Fluxion of the m Power of the Log. of the Quantity $a+x$, will $= ml$;

$\frac{n}{a+x} \times \frac{x}{a+x}$. For the flowing Quantity is

$l: a+x$, or the m Power of the Log. of

$a+x$, this therefore brought to a Power less by

Unity is $l: a+x$, and multiplied into the

Exponent m , is $ml: a+x$, and the Root of

the Quantity propos'd being $l: a+x$, whose

Fluxion is $n \dot{x} \times \frac{a+x^{n-1}}{a+x} = n \dot{x}$, 'tis plain, that

the Expression comes to what is assigned above. And in like manner for another

Example the Fluxion of the $-m$ Power of

the Log. of $a+x$, will $= -ml: a+x$

$\frac{n \dot{x}}{a+x} = -m \frac{n \dot{x}}{l: a+x} \times \frac{x}{a+x}$; for a Quantity

that of a Multiplier becomes a Divisor, has its Exponent chang'd from Positive to Negative, and *vice versa*; the reason of which we have demon-

demonstrated before. If it had been required to

find $l: x + a$, that is, the Fluxion of $l: x + a$

(instead of $l: x + a$) then the Expressi-

on of the Fluxion in that case had been $m l:$

$x + a$ $\times \frac{x}{x + a^n}$, for the Exponent n affects not (in

this Supposition, the whole Binominal $a + x$, but only the Quantity a .

I need not after the Examples that have been given, insist upon shewing, how to find the Fluxions of Expressions, where we have the Powers of Logarithms multiplied into any other ordinary flowing Quantities; since the Reader knowing how from the common Rules to find the Fluxions of any Products of flowing Quantities, and also from what has now been said, knowing how to find the Fluxions of any Powers of Logs. can't miss of knowing how to find the Fluxions of such Products, as contain the Powers of Logs. multiplied into any other flowing Quantities. From the *Lemma* foregoing we may infer as follows.

A R T.

A. R. T. IV.

C O R O L L.

The Fluxions of the Logarithms of any Powers of the same Quantity, are to one another in proportion as the Exponents of those

Powers. *Ex. gr.* $l: 1 + x: l: 1 + x^n :: 1: n$

For $l: 1 + x = \frac{x}{1 + x}$, and $l: 1 + x^n = \frac{nx}{1 + x^n}$

$\frac{nx}{1 + x^n} : \frac{x}{1 + x} = \frac{nx}{1 + x^n} \cdot \frac{1 + x}{x}$. Now 'tis manifest that

$\frac{nx}{1 + x^n} : \frac{x}{1 + x} :: 1: n$, which are the Exponents of the Powers respectively. So that the Fluxion of the Log. of the Root is $\frac{1}{2}$ the Fluxion of the Log. of the Square, and $\frac{1}{3}$ of the Fluxion of the Log. of the Cube, and so on.

Again; from hence 'tis manifest, that the Fluxions of the Logs. of any Powers of the same Quantity, are in proportion to one another, as those Logs. themselves. For the Logs. of the Root, Square, Cube, &c. are to one another as the Exponents of the Root, Square, Cube, &c. But these Exponents are as the Fluxions, as was just now shewn; and therefore the Logs. are as the Fluxions.

Lastly,

Lastly, as this *Lemma* is here demonstrated by the help of the *Logarithmick Series*, so *vice-versa*, supposing it to be demonstrated from some other Principle (as it may from a great many) the *Logarithmick Series* will be found as a consequence from hence. If, I know, *Ex. gr.* from some other

Principle, that $l: x+1 = \frac{x}{1+x}$; then 'twill

follow that $l: \frac{1}{1+x} = F. \frac{x}{1+x} = F. x \cdot \frac{1}{1+x}$

$= F. x - x \cdot \frac{1}{1+x} = x - \frac{x}{1+x}$, &c. $= x - \frac{1}{2}x^2 +$

$\frac{1}{3}x^3 - \frac{1}{4}x^4$, &c. the Log. of $1+x$.

A R T. V.

These things premis'd, there will be little difficulty in finding the *Fluxions* of *Exponential* Quantities (as they are called) or Quantities, whose Exponents are *Indeterminate* or *Flowing* ones. These sort of Quantities are of several Orders or Degrees; for when the Exponent is a simple *Indeterminate* Quantity, 'tis called an *Exponential* of the *first* or lowest Degree; when the Exponent it self is an *Exponential* of the first Degree, 'tis call'd an *Exponential* of the *second* Degree; and if the Exponent were an *Exponential* of the second Degree, it would be an *Exponential* of the third Degree; and so on. Thus, *Ex. gr.*
the

the Expression z^y is an Exponential Quantity of the first Degree, for the Exponent y is a simple flowing Quantity; and z^x is an Exponential of the second Degree, the Exponent y being an Exponential of the first Degree; and in like manner z^x is of the third Degree, the Exponent y being of the second. Two or three Examples of the Fluxions of this kind will be sufficient.

A R T. VI.

E X A M P L E I.

Let it be requir'd to find the Fluxion of the Expression a^x , where a is a constant Quantity.

Put $a^x = y$, then y will equal the Fluxion of a^x .

Now because $a^x = y$; therefore (taking the Lo-

garithms on both sides) $l: a^x = l: y$, but from

the Nature of Logs. $l: a^x = x l: a$ that is the Log. of any Power is equal to the Log. of Root multiplied into the Exponent of that Power; therefore $x l: a = l: y$. And taking

the Fluxions, we have $x l: a + x l: a = \dot{l}: y$; but

but (from what has been shewn before) $\dot{l}:a = \frac{\dot{a}}{a} = 0$, because a is constant Quantity and so $\dot{a} = 0$, and $\dot{l}:y = \frac{\dot{y}}{y}$; therefore $\dot{l}:a = \frac{\dot{y}}{y}$, or $\dot{y} = y \dot{l}:a x = a^x \dot{l}:a x$, (because $y = a^x$) which is therefore the Fluxion of a^x .

A R T. VII.

E X A M P L E II.

Let it be requir'd to find the Fluxion of the Expression z^y . Put $z = v$, then will $\dot{l}:z = \dot{l}:v$ or $y \dot{l}:z = \dot{l}:v$, and taking the Fluxions, $\dot{l}:z y + y \dot{l}:z = \dot{l}:v$; that is, $\dot{l}:z y + y \frac{\dot{z}}{z} = \frac{\dot{v}}{v} = \frac{\dot{v}}{z^y}$.

because $v = z^y$. Therefore reducing, we have $z \dot{l}:z y + y z^{y-1} \dot{z} = \dot{v}$ the Fluxion of z^y .

A R T. VIII.

E X A M P L E III.

Let it requir'd to find the Fluxion of the Expression z^x . Put $z^x = v$, then $\dot{l}:z^x = \dot{l}:v$
or

only $l:z = l:v$, and taking the Fluxions, we have $y l:z + y l:z = l:v$. But now by the last

Example, $y l:z$ that is the Fluxion of y is equal $y l:y x + x y l:y$, also $l:z = \frac{z}{z}$, and $l:v = \frac{v}{v}$; so that substituting these values in

the last Equation, we have then $l:z y l:y x + l:z x y y + y l:z = \frac{v}{v}$. From whence

multiplying each Term by z^x , we have $z^x l:z y l:y x + z^x l:z x y y + y z^{x-1} z = v$ the Fluxion of z^x .

These few Examples may be sufficient to give the Reader a taste of this Doctrine.

Note, To prevent Mistakes, the Letter l , always denote the Log. of that Quantity, 'tis Immediately set before, and not of any other that comes after.

SECT.

SECTION VII.

*Of the inverse Method, or finding the
flowing Quantities of Fluxions
propos'd.*

ART. I.

WE have hitherto been conversant about the
Direct Method of Fluxions only; that is,
how from any flowing Quantities propos'd to
find the Fluxions, it will now be necessary to add
something concerning the *Inverse Method*; that is,
how from the Fluxions propos'd to go back to
the flowing Quantities. This is vastly more dif-
ficult to do (taking the Problem in its full
scope and extent) than the former; neither is
there any general Rule that will help us in all
cases to find the flowing Quantities from the
Fluxions given, as there is to find the Fluxions
from the flowing Quantities given. The Rea-
sons of this Difficulty will appear better by
Instances and Examples, than bare Words.
However, to shew how great 'tis, 'tis sufficient
to say; that could this Problem of finding
the flowing Quantity of any Fluxion propos'd,
be

be compleatly and universally solv'd, we might find an easie Passage into the most hidden Reces-
ses of *Geometry*, and venture to pronounce the
greatest Difficulties of that Science, *No longer*
Invincible. And such a Victory once gain'd,
all the Misteries of Nature would un-fold them-
selves, of course. For if *Geometry* thus im-
prov'd, has help'd to un-riddle so many of those
Secrets to us, what thing could continue a Se-
cret, if this were brought to the desired Per-
fection? But to proceed.

A R T. II.

In the general; 'tis certain, that to find the
flowing Quantity of a Fluxionary Expression,
is to do the Reverse of that Operation, by
which the Fluxion it self was produc'd. So
that *Multiplication*; *Depressing the Powers of flow-*
ing Quantities; and putting in *Fluxionary Let-*
ters in the *Direct* Operation; are to be answer-
ed, by *Division*; *Elevation of Powers*; and stri-
king out *Fluxionary Letters*; in the *Inverse* Ope-
ration. And this *Division* in the one case is to be
made by the same Quantities as that *Multiplicati-*
on was in the other; and this raising or *Elevation*
of the Powers, to be just as much as that *De-*
pression.

A R T.

A R T. III.

But there is yet something else belonging to the Inverse Operation, to be consider'd and brought in, to the account, and that is, the *connecting of pure constant Quantities*, with the Expressions of the Fluents, found by the steps above-mention'd; the Reason of which is this. The flowing Quantity of a Fluxion propos'd may be either an Expression compos'd only of flowing Quantities, or else it may consist of flowing Quantities, with which some constant Quantities are connected by the Signs $+$ or $-$; and either way, 'tis evident that the very same Fluxion will be produc'd; concerning which we shall speak more particularly by and by. Now in the producing of a Fluxion, all *pure constant Quantities* vanish and are lost, their Fluxions being equal to nothing. And therefore, since they thus go out, leaving no mark or foot-steps behind them in the Fluxion, by which 'tis possible to know that they were really in the Expression of the Fluent; there ought to be some Provision made in the Expression of the Fluent, for the reparation of that loss, which is *at least* a possible one. For tho' in descending to particular cases; sometimes upon certain particular Suppositions made, we can be sure that there

K

are

are no invariable or constant Quantities thus to be join'd ; yet when we are proceeding in a general and abstracted way, remote from all such Considerations and Suppositions, we ought to provide for what was even possible to be, and render the Expressions we use capable of being accommodated to all cases. Now the Expressions were not universal and capable of such an Accomodation, if there were not some invariable Quantities connected with them. For in that case, they could only represent flowing Quantities that were purely and solely such ; and not such as besides, have constant Quantities join'd by the Signs $+$ and $-$ along with them. When as 'tis certain that a Fluxion propos'd may be produc'd from the one, as well as the other, of these two sorts of Expressions ; and therefore regard ought to be had to it in the Expression of the Fluent.

A R T. IV.

To illustrate this a little. Suppose the Fluxionary Expression ax ; we say, that the Fluent of this is ax , and 'tis true that it is or may be so ; but then 'tis most certain also that there are an infinite Number of Expressions besides ; each of which may be the Fluent of the same Fluxion. The Quantity ax cannot possibly have any thing else for its Fluxion but the Expression

pression ax ; but the Expression ax may possibly have some other Quantity for its Fluent besides ax : For if we imagine, $b, c, d, f, g, \&c.$ to be constant Quantities, then $ax \pm b, ax \pm c, ax \pm d, ax \pm b \pm c \pm d$: In short, $ax \pm$ an infinite Variety of other constant Quantities, may any of them be the flowing Quantity of

this Expression ax . And this being so; 'tis necessary that the Fluent should be expressed after such a general manner, that it may represent indifferently any of all those various flowing Quantities, which will return the same Fluxion

on ax , and which may be either greater or less than the Quantity ax . 'Tis upon this account that (to go genuinely and accurately to work)

the flowing Quantity of ax is not to be expressed by ax , but $ax \pm q$; where q denotes a constant Quantity, of no determinate value, but taken at liberty, and which may be equal to any other Expression whatsoever consisting of pure constant Quantities. And the like is to be understood, as to the Expression of the flowing Quantities, of any other Fluxions. If at any time they are set down without these constant Quantities connected with them, the Mathematical Reader must supply that defect in his own Mind; except where (in any particular case that occurs) 'tis certain that there are no

such Quantities to be connected, and so consequently no need of any such supply.

A R T. V.

These things premised, let the Fluxion $m x^{m-1}$ be propos'd. The flowing Quantity of this, is certain, shall be x^m . For as we come to this Expression $m x^{m-1}$, by depressing x to a Power whose Index is less by Unity, putting in the Fluxionary x , and multiplying all by the Index m ; so we come back to x^m again, by raising x^{m-1} to a Power whose Index is greater by Unity, striking out the Fluxionary x , and dividing the whole by the Index m .

So that the Rule for the Fluents of all Fluxionary Expressions, that involve some Power of a flowing Quantity, is this, *viz.* striking out the Fluxionary Letter, to add Unity to the Exponent of the flowing Quantity in the Expression, and divide, by that Exponent thus encreased by Unity.

As here in the Fluxion $m x^{m-1}$ I strike out x , and add 1 to $m-1$ (the Exponent of x in the given Expression) which makes $m x$; then I divide this by m (the Exponent encreased by 1) and

and the resulting x^m is the flowing Quantity required. Or if the Fluxion had been x^{m-1} , then x^m must have been divided by m , and so the Fluent would have been $\frac{1}{m} x^m$.

ART. VI.

Thus also the Fluent of $\frac{n}{m} y^{\frac{n-m}{m}}$ is $y^{\frac{n}{m}}$; of $-ny^{\frac{-n-1}{m}}$, is $y^{\frac{-n}{m}} = \frac{1}{y^{\frac{n}{m}}}$; of $\frac{1-n}{x x^{\frac{n}{m}}}$, is $n x^{\frac{1-n}{m}}$; of $\frac{1}{n} x^{\frac{1-n}{m}}$, is $x^{\frac{1}{m}}$ of $\frac{x^n}{p x^{\frac{1}{m}}}$ ($= \frac{m-1}{p} x^{\frac{m-1}{m}}$) is $\frac{m x^{\frac{m-1}{m}}}{p}$; of $x x^{\frac{n}{m}} \sqrt{p x}$ ($= x x^{\frac{n}{m}} p^{\frac{1}{2}} x^{\frac{1}{2}}$) is $\frac{m p + m p - p}{2 n + 1} x^{\frac{2n+3}{2}} p^{\frac{1}{2}}$. When the

Exponents are express'd by Numbers of any sort, it will be easie from what has been said, to assign the Fluents. Thus the Fluent of

$2xx$, is xx ; of $3xxx$, is x^3 , of $4x^3 x$, is x^4 , of xx , is $\frac{1}{2} xx$, of xxx , is $\frac{1}{3} x^3$; of $\frac{1}{2} x x^2$, is x^3 ; of $\frac{1}{2} x^{\frac{1}{2}} x$, is $x^{\frac{3}{2}}$; of $\frac{1}{2} x^{\frac{1}{2}} x^{\frac{1}{2}}$, is $x^{\frac{1}{2}}$; of $\frac{1}{12} x^{\frac{1}{2}} x^{\frac{1}{2}}$, is $x^{\frac{1}{2}}$.

x , is $\frac{2}{3} x^{\frac{2}{3}}$; and the like in other Cases; for these Examples need not be multiplied upon the Reader.

A R T. VII.

From the same Rule are found the Fluents of any compound Expressions; where the Terms consist of any Powers of flowing Quantities multiplied into their respective Fluxions. Thus

Ex.gr. the Fluent of the Expression $a^3 x x + b^3 x z + y^4 y$, is $\frac{3}{2} a^3 x^2 + \frac{1}{4} b^3 x z + \frac{5}{5} y^5$. And so of others of the like nature. There's nothing more to be done in this case, but to *connect the Fluents of the several Terms together, with their proper Signs.*

A R T. VIII.

As for the Fluents of such Expressions, where there are no Powers of any flowing Quantities, but Products of flowing Quantities multiplied into Fluxions, and also the Fluxion of every flowing Quantity that is in the Expression: These are easily found by the *Reverse of the Direct Operation.*

Suppose we had the Fluxion $y x + x y$; the Fluent of this Expression will be xy , For as the

the Expression $y\dot{x} + x\dot{y}$ is form'd by setting down each side (of the Rectangle xy) severally multiplied into the Fluxion of the other side; of one Term thereby making two; so by substituting (instead of the Fluxions $\dot{x}$ and $\dot{y}$) the respectively flowing Quantities x and y , and of both the Terms making but one; that is, dividing the Sum of the Terms by the Number of the Terms; by this Operation the Expression xy it self is produc'd. So will the Fluent of $xyz + zxy + zyx$ be found to be xyz ; the flowing Quantities x, y, z , being substituted in the places of Fluxions $\dot{x}, \dot{y}, \dot{z}$, and bringing the three Terms or Members by Division into one. And in the like manner, for other such Expressions as these are, where there is found the Fluxion of every particular Quantity that is in the Expression, and no Fluxion is multiplied into its own proper flowing Quantity. As in this last Expression where there are three flowing Quantities, x, y, z , as also each of their Fluxions $\dot{x}, \dot{y}, \dot{z}$, and $\dot{z}$ is not found in the same Term with z , nor $\dot{y}$ with y , nor $\dot{x}$ with x . The Rule in all such cases, I say, is to substitute instead of each Fluxion its respective flowing Quantity, and adding all the Terms together to divide that Sum by the number

K 4

ber of the Terms. As here in the Fluxion xyz
 $+ zxy + zyx$, putting z, y, x , instead of $z,$
 y, x , we thereby makes $3xyz$, which divided
 by 3, the number of the Terms, gives xyz the
 Fluent of the Expression given.

A R T. IX.

When Fluxional Expressions are propos'd
 that are affected with a *Vinculum*, that is, where
 there is some compound surd Quantity multi-
 plied into a Fluxion: We are to consider whe-
 ther the Fluxion standing before the Radical
 Sign be the Fluxion of the Quantity contain'd
 under the Radical Sign. For if it be, then the
 Fluent may be found by the general Rule for
 Powers before given.

Thus, *Ex. gr.* if the Expression $ax\sqrt{ax - aa}$
 were given; here the Quantity under the *Vin-*
culum or Radical Sign, is $ax - aa$, and the
 Fluxion of that (by the *direct Method*) is ax ,
 the Quantity a being constant. So that in this
 case, the Fluxion without the *Vinculum*, viz.
 ax is the Fluxion of the Quantity contain'd
 under the *Vinculum*, and consequently the com-
 mon Rule will give the Fluent. That is, we
 must

must raise $\sqrt{ax-aa}$, or $ax-aa$ to a Power whose Exponent is greater by Unity, and divide by the Exponent of the Power thus en-

creas'd. And then the Fluent of $ax \sqrt{ax-aa}$

or $ax \times ax-aa$, will be $\frac{2}{3} ax-aa$, or $\frac{2}{3}$

$ax-aa \sqrt{ax-aa}$. For if we add 1 or $\frac{1}{2}$ to $\frac{1}{2}$ which is the Exponent of $\sqrt{ax-aa}$, we have then $\frac{3}{2}$ for the Exponent of the new Power; and

A R T. X.

Other Examples of the same kind, are also such as follow. The Fluent of $xx + ax \sqrt{a+x}$,

that is of $x \sqrt{a+x}$, or of $x \times a+x$; is $\frac{2}{5}$

$a+x = \frac{2xx + 4ax + 2aa}{5} \sqrt{a+x}$. The

Fluent of $x \times a+x$, is $\frac{m}{m+n} a + \frac{n+m}{x^m}$.

The Fluent of $2xx \sqrt{xx+aa}$, that is of

$2xx \times xx+aa$, is $\frac{2}{3} xx+aa = \frac{2xx+2aa}{3}$

$\sqrt{xx+aa}$. The

The Fluent of $p x^{\frac{p-1}{n+1}} \sqrt[n]{x^n + a^n}$, is $\frac{1}{n+1} \sqrt[n+1]{x^{n+1} + a^{n+1}}$. And the like of any other Examples, where the Fluxional Expressions are so qualified as to fall within the compass of this Rule.

A R T. XI.

But to shew more effectually, how all Examples of this kind are done entirely by the general Rules for Powers before deliver'd ; we may go thus to work, taking the Quantities out of those *Surd* Expressions, and putting them into *Rational* ones, by a concise and commodious Substitution. *Ex. gr.* Suppose the Expression $2xx\sqrt{xx+aa}$, were given. Put $xx+aa=z^2$; then will $\sqrt{xx+aa}=z$, and $2xx=2zz$, and so $2xx\sqrt{xx+aa}=2zzz$; and therefore the flowing Quantity will $=\frac{2}{3}z^3$, which is equal by Substitution) to $\frac{2}{3}\sqrt[3]{xx+aa}=\frac{2xx+2aa}{3}\sqrt{xx+aa}$, as was found before.

A R T.

A R T. XII.

If the Fluxionary Expression without the *Vinculum*, be not the Fluxion of the Quantity contain'd under the *Vinculum*, but yet is in some given Ratio to it; the Expression of the Fluent may, notwithstanding, be had in Finite Terms.

Thus if the Expression $xx\sqrt{xx+aa}$ were given, where xx the Fluxion without the *Vinculum*, is to the Fluxion of the Quantity contain'd under it (*viz.* $2xx$) as 1 to 2. Put (as

before) $\sqrt{xx+aa} = z$, then $2zz = 2xx$, or

$xx = zz$, and so $xx\sqrt{xx+aa} = zz$, and

consequently the flowing Quantity $= \frac{1}{3}z^3 =$

(by substitution) $\frac{xx+aa}{3}\sqrt{xx+aa}$. Again,

suppose we had this Expression $x^{\frac{m-1}{n}}x^m+aa$.

Here the Fluxion without the *Vinculum*, *viz.*

$x^{\frac{m-1}{n}}$ is to the Fluxion of the Quantity con-

tain'd under it, *viz.* $\frac{m-1}{n}x^{\frac{m-1}{n}}$, as 1 to $\frac{m-1}{n}$; which

is a given Ratio. Put $x^{\frac{m-1}{n}}+aa = z$, then

$x^m+aa = z^n$, and $x^m+aa = z^n$; also

(by

(by the *Direct Method*) $nz^{\frac{n-1}{m}} = nm x^{\frac{m-1}{m}} z^{\frac{n-1}{m}}$
 $x x^{\frac{n-1}{m}} + a^{\frac{1}{m}} = nm x^{\frac{m-1}{m}} z^{\frac{n-1}{m}}$; whence divi-
 ding by $nz^{\frac{n-1}{m}}$, we have $z = m x^{\frac{m-1}{m}} z^{\frac{n-1}{m}}$, or
 $x^{\frac{n-1}{m}} = \frac{z}{m}$. Therefore, now by these substi-
 tutions, we have $x^{\frac{m-1}{m}} x x^{\frac{n}{m}} + a^{\frac{1}{m}} =$
 $\frac{z^{\frac{n}{m}}}{m}$; and the flowing Quantity = $\frac{I}{nm + m}$
 $z^{\frac{n+1}{m}} = \frac{I}{nm + m} x^{\frac{n+1}{m}} + a^{\frac{1}{m}}$.

A R T. XIII.

If the Fluxion without the *Vinculum*, be
 neither the Fluxion of the Quantity contain'd
 under it; nor bears not any given Ratio there-
 to; yet notwithstanding in many forms of Ex-
 pressions, upon certain suppositions made, the
 Fluents may be had in Finite Terms, tho' not
 by the general Rule, as in the first case.

Ex. gr. Let the Fluxion $x^{\frac{p}{n}} \sqrt[n]{a+x}$, or
 $x^{\frac{p}{n}} x^{\frac{1}{n}} a^{\frac{1}{n}} + x^{\frac{p}{n}} x^{\frac{1}{n}}$, be propos'd where $x^{\frac{p}{n}} : x^{\frac{1}{n}} :: x : 1$.
 and consequently not in a given Ratio. If the
 Exponent p be an Integer, (which is supposed to
 be given in particular Cases) the Fluent of the
 Expression may be had in Finite Terms, but
 will

will consist of a greater or less number of Terms according as p is greater or less: Put $a + x =$

z ; then it follows that $z = x$, $z - a = x$,

$$\frac{p}{x-a} = \frac{p}{x}, \quad \frac{1}{x+a} = \frac{1}{x}. \quad \text{Therefore } x \neq x.$$
$$a + x = z \cdot z \cdot z \cdot \dots \cdot z - a. \text{ But } z - a = z - pz$$

+ $\frac{pp - p_z^{p-2}}{2} aa$, &c. which Series will

terminate when a number equal to p comes to

be subtracted from it. So that $x^{\frac{1}{n}} \times \overline{x - a^p} =$

$$\frac{np+1}{2n} \dot{x} - \frac{np-n+1}{2} a + \frac{np-2n+1}{2} \dot{x} \frac{1}{na} \mathcal{O}c$$

And the flowing Quantity, by the common

Rule, is equal to $\frac{n}{np + n + 1} \approx \frac{np + n + 1}{n}$ —

$$\frac{np}{np+1} \approx \frac{np+1}{n} \approx \frac{npp-np}{2np-2n+2} \approx \frac{np-n+1}{n} \approx \dots$$

In which Series, if instead of z we substitute it equal $a+x$, we have then the Expression of the Fluent in the Terms of the Quantity propos'd and those finite in number too, as is most evident.

The Reader will easily accommodate this Series to any particular Cases, by expounding the Quantities p or n , of any numbers at Liberty.

ART.

S C H O L.

The Fluents in such Instances as these, are obtained in finite Terms ; but (as I said) *not by the general Rule, as in the first Case.* For there the Fluxion without the *Vinculum* being the very Fluxion of the Quantity contained under the *Vinculum*, the Fluent comes out in one entire Expression; purely and solely by vertue of that Rule. But here, though the several Members or Parts of the Fluents are found by that same Rule too, yet the *Sum* or *Complex* (of all those Members or Parts) *consider'd as one entire Quantity*, is not gotten or deduced from that Rule; the conditions of the Fluxional Expressions here being evidently such, as in their own Nature will not allow it. And the same is to be said of the Case immediately foregoing, when the Fluxion without the *Vinculum* is in a given *Ratio*, to the Fluxion of the Quantity contained under it. Yet in that case, the Fluent may be had in one compleat Expression ; whereas, in this case, it will ever consist of several Members, though the number of them will still be finite.

ART. XIV.

Supposing all as in the last *Art.* only now let the flowing part of the Quantity contain'd under the *Vinculum*, be rais'd to some Power, which there it was not : and then also, in many cases, may the Expression of the Fluent be had in finite Terms.

Ex. gr. Suppose we had this Expression $x^p \cdot \frac{1}{n}$
 $x a a + x x$, propos'd, where the flowing part of the Quantity under the *Vinculum*, is a Power, viz. $x x$. Put $a a + x x = z$; then it follows,

that $z = 2 x x$, $\sqrt{z} = a a$, or $z - a a = x x$,

$\frac{z a - a a}{2 x} = x^p$, $a a + x x = z$. Also $x =$

$\frac{z}{2 x} = \frac{z}{2 \sqrt{z a - a a}}$. Therefore it follows,

that the Expression $x^p \cdot \frac{1}{n}$
 $x x \cdot a a + x x$, is (in the Terms of z) $= \frac{z^{\frac{1}{n}} \cdot z - a a^{\frac{p}{n}}}{2 \sqrt{z - a a}}$, which (by

the Rule for the Division of Powers) comes to

$\frac{1}{n} \cdot \frac{p-1}{2} z^{\frac{1}{n}} z^{\frac{p-1}{n}} - a a^{\frac{p-1}{n}}$. Now if the Exponent $p-1$ be an Integer, which will be when p is an

odd number, the Quantity $z - a a$ rais'd to that

that Power will consist of a finite number of Terms, and so the Expression of the Fluent will be had in Finite Terms also. But if it be a Fraction, the Fluent will be express'd by an infinite Series, which is therefore the last help we can have recourse too.

A R T. XV.

If the flowing part of the Quantity contain'd under the *Vinculum* be rais'd to some Power, and the Fluxional Expression without the *Vinculum* has no Power of the Flowing Quantity in it, but be only the Fluxion of the Root; in this Case the Expression of the Fluent will always run up in an infinite Series.

Ex. gr. Suppose the Fluxion $x \cdot a a + x x$, were given, where the flowing part *within* the *Vinculum* is a Power of x , and the Fluxion *without*, is only the Fluxion of the Root. Put (as before) $a a + x x = z$, and consequently

$$\dot{x} = \frac{\dot{z}}{2\sqrt{z - a a}}, \text{ and therefore } x \cdot a a + x x = z z^{\frac{1}{2}}$$

$$= z z^{\frac{1}{2}}. \text{ Now here's no remedy, but}$$

we must have recourse to an infinite Series for the

the Expression of the Fluent of $z z^{\frac{1}{n}}$ $\frac{1}{2\sqrt{z-aa}}$, or

$z z^{\frac{1}{n}} \times \frac{1}{2\sqrt{z-aa}}$ (which is all one) $z z^{\frac{1}{n}} \times 2 z^{\frac{1}{2}} \sqrt{z-aa}$. Whereas had the form of the Fluxion been as 'twas in the last Article, we had then had some Power of the Quantity $z-aa$ both in the Numerator and Denominator; which, therefore, had left us also some other Power (of that same Quantity $z-aa$) multiplied into $z z^{\frac{1}{n}}$, and consequently upon some particular Determinations of the Exponent of that Power, we had obtain'd the flowing Quantity in finite Terms.

But here, we have only some Power of $z-aa$ in the Denominator, and that is, the $\frac{1}{2}$ Power or Square Root of it; so that that Reduction can never take place here that did there, and consequently the Expression of the Fluent can never be had in finite Terms. So that all that can be done in this Case, is to raise the Quantity $z-aa$ to the Power whose Exponent is $-\frac{1}{2}$, which will be an infinite Series, and then multiply $z z^{\frac{1}{n}}$ into each Term of that Series; and, lastly (by the common Rules) take the flowing Quantity of every one of those Terms; for the Sum of all will give the Expression of the Fluent sought. But this Process shall be more particularly illustrated by the following Example.

L

A R T.

ART. XVI.

Let it be required to find the flowing Quantity of the Expression $x\sqrt{rr-xx}$. Here first we must elevate $rr-xx$ to the $\frac{1}{2}$ Power, which will be an infinite Series. viz. $rr-xx^{\frac{1}{2}} = r - \frac{x^2}{2r} - \frac{x^4}{8r^3} - \frac{x^6}{16r^5} - \frac{5x^8}{128r^7} - \&c.$ Then multiplying x into this Series, and taking the Flu-ent of each Term, we have $rx - \frac{x^3}{6r} - \frac{x^5}{40r^3} - \frac{x^7}{112r^5} - \frac{5x^9}{1152r^7} - \&c.$ for the Fluent of the Expression $x\sqrt{rr-xx}$. And so in the same manner for other Examples of the like nature and condition.

I have now in these few things, said as much concerning the Practise of the Inverse Method as is sufficient for my present purpose and design. I shall now only subjoin some few Rules, which may help to direct the young Geometer in his Reasonings about Fluxions and flowing Quantities.

S E C T.

S E C T. VIII.

A R T. I.

If any two different Powers of different flowing Quantities, be reciprocally, as their Fluxions, then those Powers shall be equal.

Let $x : y :: ny^{\frac{n-1}{n}} : mx^{\frac{m-1}{m}}$; then making an Equation; from the Rule of the Multiplication of Powers, it will be $mx^{\frac{2m-1}{m}} = ny^{\frac{2n-1}{n}}$. Consequently by the inverse Method $\frac{m}{2m} x^{\frac{2m}{m}} = \frac{n}{2n} y^{\frac{2n}{n}}$, that is, by equal Division and Extraction, $x^m = y^n$.

A R T. II.

If any two, the same Powers, of two flowing Quantities, be reciprocally as the Fluxions of the Roots, those Quantities are equal, and their Fluxions also.

If $x : y :: y : x$, then $x^m = y^m$, and by the Inverse Method, $\frac{1}{m+1} x^{m+1} = \frac{1}{m+1} y^{m+1}$, and consequently $x=y$. From whence again by the direct Method, it follows, that $x=y$.

A R T. III.

If two flowing Quantities are not in a constant Ratio, their Fluxions cannot possibly be in a constant Ratio.

Let z be a flowing and d a constant Quantity;

and let $x : y :: z : d$, then 'tis plain that

x and y are not in a constant Ratio; for the Ratio of the flowing z , to the constant d , is a flowing or mutable Ratio. Now taking a, b any two constant Quantities: I say, 'tis impossible,

that $m x^{m-1} : n y^{n-1} :: a : b$. For suppose it be so. Then $b m x^{m-1} = a n y^{n-1}$, and

consequently (by the Inverse Method) $b x^m = a y^n$, therefore $x : y :: a : b$. But $x : y :: z : d$;

therefore $a : b :: z : d$, and so z to d , is in a constant Ratio, which is absurd. Therefore the Supposition is so also.

N. B. In these three foregoing Rules, we have inferr'd the Equality of the Fluents, from that of the Fluxions, without any regard to constant Quantities, which the young Geometer may think ought to be connected with the Fluents, considering what was said before upon that Subject. Thus at Rule the first, because

'twas $m x^{\frac{2m-1}{2n-1}} = n y^{\frac{2n-1}{2m-1}}$, therefore 'twas concluded that $\frac{m}{2m} x^{\frac{2n}{2n-1}} = \frac{n}{2n} y^{\frac{2m}{2m-1}}$; not $\frac{m}{2m} x^{\frac{2m}{2m-1}} + q = \frac{n}{2n} y^{\frac{2n}{2n-1}} + g$; which, had it been so, we could

not have inferr'd that $x^{\frac{m}{n}} = y^{\frac{n}{m}}$. But let it be consider'd; that by our Hypothesis, the Fluents of $m x^{\frac{m-1}{n}}$ and $n y^{\frac{n-1}{m}}$, were the Expressions $x^{\frac{m}{n}}$ and $y^{\frac{n}{m}}$ alone without any constant

Quantities. Now if the Fluent of $m x^{\frac{m-1}{n}}$ (Ex. gr.) be suppos'd to be $x^{\frac{m}{n}}$ without constant

Quantities, the Fluent of $m x^{\frac{m-1}{n}}$ taken any number of times (whether a determinate, or an indeterminate number) must according to that

supposition be still some expression of $x^{\frac{m}{n}}$ without constant Quantities. So that therefore the Fluent of x times $m x^{\frac{m-1}{n}}$, or of $x^{\frac{m}{n}} \times m x^{\frac{m-1}{n}}$ that

is, of $m x^{\frac{2m-1}{2}}$, ought to be taken an Expression without constant Quantities. And in like manner for the Fluent of the Expression $n y^{\frac{2n-1}{2}}$. And 'twill appear by the same way of reasoning in the other two Rules also, that no constant Quantities are to be suppos'd to be connected with the Expressions of the Fluents.

A R T. IV.

If two flowing Quantities are in a given Ratio, their Fluxions shall be also to one another, in that same given Ratio; but the Reverse of this will not follow, universally speaking; for if two Fluxions propos'd are in any given Ratio, the Fluents either may be in that same given Ratio, or they may possibly not be so. Let a and b be standing Quantities, x and y flowing, and suppose $x : y :: a : b$. Then $x^m b = y^n a$, and (by the Direct Method) $b m x^{\frac{m-1}{2}} = a n y^{\frac{n-1}{2}}$, consequently $m x^{\frac{m-1}{2}} : n y^{\frac{n-1}{2}} :: a : b$; that is, the Fluxions of x and y , are as a and b .

But now for the Reverse of this Rule: Suppose the two Fluxions $m x^{\frac{m-1}{2}}$ and $n y^{\frac{n-1}{2}}$ were given; I say, tho' it be $m x^{\frac{m-1}{2}} : n y^{\frac{n-1}{2}} :: a : b$, yet

yet it will not follow that the Fluents are
 also as $a : b$. For, tho'tis true, that $b^{\frac{m-1}{m}} x^{\frac{n-1}{n}} =$
 $a^{\frac{m-1}{m}} y^{\frac{n-1}{n}}$, and so by the inverse Method (if we
 were sure that x and y were the Fluents) $b^{\frac{m}{m-1}} x^{\frac{n}{n-1}}$
 $= a^{\frac{m}{m-1}} y^{\frac{n}{n-1}}$, from whence $x : y :: a : b$; yet 'tis
 possible that x and y may not be the Fluents
 of these Fluxions, but $x \pm g$, and $y \pm b$ (where
 g and b are constant Quantities) may be the
 Fluents, as has been shewn before. Now the
 proportion $m^{\frac{m-1}{m}} x^{\frac{n-1}{n}} : n^{\frac{n-1}{n}} y^{\frac{m-1}{m}} :: a : b$, will never
 infer that $x \pm g : y \pm b :: a : b$, as is most mani-
 fest. So that the reason of the Limitation is
 clear.

A R T. V.

If there be any flowing Quantities proporti-
 onal to one another, it cannot be inferr'd (bare-
 ly from thence) that their Fluxions shall be
 proportional to one another; or contrari-wise, if
 the Fluxions are proportional, it won't follow
 that the Fluents are so.

Ex. gr. if it be $x : y :: z : v$ I say, it will not
 follow, without some other intervening Sup-
 positions

positions, that the Fluxions of these are also pro-

portional; or that $m x^{\overline{m-1}} : n y^{\overline{n-1}} :: z : v$. For if $x :$

$y :: z : v$, then 'tis true indeed that $x^{\overline{m}} v = z y^{\overline{n}}$; and

'twill follow from thence, that by the direct Me-

thod $m x^{\overline{m-1}} v + x^{\overline{m}} v = n y^{\overline{n-1}} z + y^{\overline{n}} z$. But then

this is all that will follow, from the Analogy

suppos'd; and 'tis plain that this will never

allow a Man to infer that $m x^{\overline{m-1}} : n y^{\overline{n-1}} ::$

$z : v$. So that 'twill never follow that the

Fluxions are proportional because the flowing

Quantities themselves are so. But now if these

four flowing Quantities had been proportional

to one another, *as a consequence of their being*

proportional (taken two and two) to some standing

Quantities, we might then fairly have conclu-

ded their Fluxions to be proportional also.

As if it had been $x : y :: a : b :: z : v$; or if

$x : y :: a : b$; and $z : v :: c : d$, and it were $a :$

$b :: c : d$, then it will be in either case $m x^{\overline{m-1}} :$

$n y^{\overline{n-1}} :: z : v$, as is easily made out from the

last Rule, and the bare equality of Ratio's. The

Reverse of this Rule, is true too, viz. that the

bare proportionality of the Fluxions, will ne-

ver infer the Fluents to be proportional: That

is, it will never follow, that $x : y :: z : v$,
 from hence 'tis granted that $m x^{\frac{m-1}{x}} : n y^{\frac{n-1}{y}} :: z :$
 v ; even supposing the former Terms to be the
 Fluents of the latter, which yet may not be
 neither.

A R T. VI.

If any two flowing Quantities are in a given
 Ratio, it may be inferr'd that they are directly
 proportional to their Fluxions: But it does not
 follow, if the Fluxions are proportional to the
 flowing Quantities; that therefore those flow-
 ing Quantities are in a given Ratio.

Let $x : y :: a : b$. then it follows from thence

(by the first Rule) that $m x^{\frac{m-1}{x}} : n y^{\frac{n-1}{y}} :: a : b$ and
 consequently that $x : y :: m x^{\frac{m-1}{x}} : n y^{\frac{n-1}{y}}$, be-
 cause both are in the same Ratio of $a : b$.

But now if this proportion $x : y :: m x^{\frac{m-1}{x}} :$
 $n y^{\frac{n-1}{y}}$, be suppos'd or granted, all that will
 follow immediately from hence, is that $x^{\frac{m}{m-1}}$
 $n y^{\frac{n}{n-1}} = y^{\frac{n}{n-1}} m x^{\frac{m}{m-1}}$; that is, (dividing both
 sides by $x^{\frac{m}{m-1}} y^{\frac{n}{n-1}}$) that $n y = m y x^{\frac{m}{m-1}}$, or
 $n y = m y x^{\frac{m}{m-1}}$

$xyx = myx$, or that $x:y::mx:ny$. Now 'tis plain, this is infinitely remote from any

such consequence as this, that $x:y$, in a given Ratio. And, therefore, I say, from the proportionality of the Fluents to the Fluxions it will never follow, that the Fluents are in a given Ratio; tho' if the Fluents are in a given Ratio, that proportionality will certainly follow.

A R T. VII.

The reasonings (at Rule the 4th foregoing) about *First* Fluxions, hold in *Second*, *Third* and all the following Fluxions: If the flowing Quantities are in a given Ratio, their *First*, *Second*, *Third*, &c: Fluxions, shall be in the same given Ratio, and consequently proportional to one another, and to their respective Fluents.

Thus if $x:y::a:b$, then $mx^{\frac{m}{n}}:ny^{\frac{n}{n}}::a:b$; then these Fluxions consider'd as Fluents, their Fluxions also shall be in the same Ratio

(by that same Rule) that is, $mx^{\frac{m-1}{n}}:ny^{\frac{n-1}{n}}::a:b$; and
 $x^{\frac{m-2}{n}}:ny^{\frac{n-2}{n}}::a:b$; and
 consequently $mx^{\frac{m-1}{n}}:ny^{\frac{n-1}{n}}::mx^{\frac{m-1}{n}}:ny^{\frac{n-1}{n}}$

$$\frac{m-2}{m} x^2 : \frac{n-1}{n} y + \frac{n-2}{n} y^2 :: x :$$

y . And in like manner we may demonstrate it of the other kinds of Fluxions.

A R T. VIII.

If a constant Quantity be a third or fourth proportional to two or three flowing Quantities ; one of which flows uniformly, none of the Fluxions of those flowing Quantities, can possibly be in a *given Ratio* to each other. Suppose this Equation $d x = z y$, or (which is all

one) this Analogy, $x : y :: z : d$, where d is a standing Quantity, and let z flow uniformly. 'Tis plain that the Ratio of z to d cannot be a standing or given one ; and since $x : y :: z : d$,

it follows that the Ratio of x to y can never be a constant Ratio : Therefore the Ratio of $m x$ to $n y$, cannot be a given Ratio neither, by what was shewn at Rule the 3d.

Again, $x : z :: y : d$; and the Ratio of y to d cannot be constant (for the same reason as

that of z to d) therefore the Ratio of x to z (which

(which is the same with that of y^n to d) cannot be constant, and consequently the Ratio of $m-1$
 $m \times \dot{x}$ to $\dot{z}$, cannot be a given Ratio neither. Lastly, because z flows uniformly, and so $\dot{z}$ is constant; therefore the Ratio of $n-1$
 z to $n\dot{y}$, cannot be constant. So that in *this Hypothesis*, none of all the Fluxions of these flowing Quantities, can possibly be in any given Ratio to each other. I say, in *this Hypothesis*: where one of the 4 proportional Terms is a standing Quantity; for if all the 4 Terms had been flowing Quantities, it were possible that they might be in some given Ratio to each other, and so their Fluxions might possibly be in some given Ratio likewise. But where one of the Terms is a standing Quantity, it's utterly impossible that there should be any constant Ratio between the Fluxions of those Quantities.

A R T. IX.

If there be a Fluxional Expression in which is found a flowing Quantity, different from that whose Fluxion is there; then the Fluent of the whole expression must necessarily be less, than it would be, were that different flowing Quantity a constant one. Suppose we had the expression

pression xy ; if x were a standing Quantity, then the Fluent of the given expression would

be $\frac{xyy}{2}$, by the common Rules. But x being

a flowing Quantity, the Fluent ('tis manifest)

cannot be equal to $\frac{xyy}{2}$; and, I say, farther,

that it shall certainly be less than $\frac{xyy}{2}$, and not

greater. The Reason of which is plainly this.

Since the Expression $\frac{xyy}{2}$ consists of two flow-

ing Quantities, the Fluxion of it must neces-

sarily consist of two Members; one wherein

is the Fluxion of x , and the other in which is

the Fluxion of y : And 'tis certain (by the *direct Method*) that these two Members shall be

$\frac{yy\dot{x}}{2}$, and $xy\dot{y}$. Now then since the Fluent

$\frac{xyy}{2}$ brings for its Fluxion these two Members

$\frac{yy\dot{x}}{2} + xy\dot{y}$; of which the latter, *viz.* $xy\dot{y}$ is

the Fluxional Expression propos'd; 'tis most

evident that the Fluent of $xy\dot{y}$ must needs be

less than $\frac{xyy}{2}$, since the Fluent of $\frac{yy\dot{x}}{2} + xy\dot{y}$

is it self but equal, to $\frac{xyy}{2}$. Therefore the

Fluent

Fluent of $xy\dot{y}$ is $= \frac{x\dot{y}y}{2} - s$ (Ex. gr.) which Quantity s must necessarily be a flowing Quantity, and cannot be a standing one; for if it were supposed to be so, there would be this absurdity consequent upon it, that $xy\dot{y}$ would be equal to it self added to another Fluxion, that is, it would be greater than it self: And this easily appears; for since $F. xy\dot{y} = \frac{x\dot{y}y}{2} - s$ (as has been demonstrated;) therefore, by the direct Method, we have $xy\dot{y} = \frac{x\dot{y}y}{2} + xy\dot{y}$, if s be a standing Quantity, and $xy\dot{y} = \frac{x\dot{y}y}{2} + xy\dot{y} - s$; if s be a flowing Quantity. Now the former of these Equations is manifestly impossible for in that case either $\frac{x\dot{y}y}{2}$ is equal to nothing, or else (as we said) $xy\dot{y}$ is greater than it self; for 'tis on one side of the Equation alone, and on the other 'tis added to something else. Therefore s cannot be a standing Quantity, but is a flowing one.

In like manner, $F. zxy\dot{y} < \frac{zx\dot{y}y}{2}$, and $F. vx\dot{x} < \frac{v\dot{x}x}{2}$, and $F. z^m x^n y^p < \frac{1}{p+1} z^m x^n y^{p+1}$; and so in other cases.

A R T. X.

Where we have a Fluxional Expression in the form of a Fraction, the Fluent of the whole Expression may be either *Finite*, *Infinite*, or *more than Infinite*, according as the Exponents of the flowing Quantities, in the Numerator and Denominator, are proportion'd to one an-

other. *Ex. gr.* $F. \frac{x^{\dot{x}}}{x^m}$, will be Finite, Infinite, or more than Infinite, according as n is $>$, $=$, or $<$ than m . For because from the Rules of

Exponents, $\frac{x^{\dot{x}}}{x^m} = x^{n-m}$, therefore $F. \frac{x^{\dot{x}}}{x^m} =$

$F. x^{\dot{x}^{n-m-1}}$ $= \frac{1}{n-m} x^{n-m}$. Now if $n = m$, then

$n - m = 0$, and so $\frac{1}{n-m} x^{n-m} = \frac{1}{0} x^0 = \frac{1}{0}$,

which is Infinite. If $n > m$, then $n - m =$ some positive Quantity as q , and so $\frac{1}{n-m} x^{n-m}$

$= \frac{1}{q} x^q$, which is Finite. If $n < m$, then $n - m =$ some Negative Quantity, as $-q$, and so

$\frac{1}{n-m} x^{n-m} = \frac{1}{-q} x^{-q} = \frac{1}{-qx^q}$, which is more than Infinite.

If

If the Exponent m in the Denominator were Negative, the Exponent n in the Numerator continuing positive, then the Fluent of the Expression would still be Finite.

If the Exponent n in the Numerator were Negative, and the Exponent m in the Denominator were positive, the Fluent would still be more than Infinite.

If the Exponents m and n were both Negative, the Fluent would be Finite, Infinite or more than Infinite, according as m was $>$, $=$, or $<$ than n .

The like also is to be observed in such Expressions as these, viz. $F. \frac{\dot{x}}{x^n}$, where the Fluent will also be Finite, Infinite, or more than Infinite, according as m is $<$, or $=$, or $>$ than 1.

So that to instance particularly, $F. \frac{\dot{x}}{x^2}$, $F. \frac{\dot{x}}{x^3}$,

$F. \frac{\dot{x}}{x^4}$, &c. are Finite ; $F. \frac{\dot{x}}{x}$, is Infinite ; $F.$

$\frac{\dot{x}}{x^2}$, $F. \frac{\dot{x}}{x^3}$, $F. \frac{\dot{x}}{x^4}$, &c. are more than Infinite ;

and so of others.

If the Expressions were such as these, viz. $F.$

$\frac{x^{n-m}}{x^{p-q}}$, or $F. \frac{x^{n+m}}{x^{p-q}}$, or $F. \frac{x^{n-m}}{x^{p-q}}$, &c. where

the Exponents of the Quantities in the Numerator

erator and Denominator consist of more than one Term, and are variously affected the Signs + and —; we may determine in like manner; when the Fluent shall be Finite, Infinite, or more than Infinite; tho' the Determination will be something more compounded, than where the Exponents are expressed in simple Quantities.

$$\text{Thus, ex. gr. } F. \frac{x^{n-m}}{x^{p-q}}, = F. x^{n-m-p+q} =$$

$$\frac{1}{n-m-p+q+1} x^{n-m-p+q+1}; \quad \text{is Infinite when}$$

$n+q+1 = m+p$, is Finite when $n+q+1 > m+p$, is more than Infinite, when $n+q+1 < m+p$. Also it is more than Infinite, when $n+q+1 = p$, or $n+q+1 = m$; and it is Finite when $n+q+1 > m$, and the Excess is $> p$; or when $n+q+1 > p$, and the Excess is $> m$. Again; it is Finite when $n+q > m+p$, or $n+1 > m+p$; and 'tis more than Infinite when $m+p > n+q$, and the Excess is > 1 , or when $m+p > n+1$, and the Excess is $> q$. Again; 'tis Infinite when $n = m$, and $p = q+1$; or when $n = p$, and $m = q+1$; or when $m+p > n+q$, and the Excess is $= 1$; or when $m+p > n+1$, and the Excess is $= q$. And thus were the Exponents still more compounded, there would be still more Varieties in the Determinations;

SECTION IX.

WE have now given an account of those Rules in the *Direct* and *Inverse* Method of Fluxions that are *Fundamental*, and ought to be expected in an Introduction. What remains now to be done, is to give the young Geometer an account of the several uses and applications of this incomparable Method, and then some few Instances and Examples of it.

A R T. I.

As the Method of Fluxions it self is distinguished into these two Parts, the *Direct*, and the *Inverse Methods*; so have each of these Parts or Branches, their several peculiar Uses and Applications.

Of the many noble Problems in Geometry, whose Solutions are easily attained to by this admirable sort of Computation; some are to be done by the *Direct*, and others by the *Inverse* Method of Fluxions. That is some Problems, do in order to their Solution, require us to find the Fluxions of certain flowing Quantities; and others makes it necessary to find

find the flowing Quantities of some Fluxions propos'd. So as that upon this finding of the Fluxion on the one hand, or the flowing Quantity on the other, lies the main stress of the Solution. And this arises from the very nature of the thing it self; and that the obtaining of what we desire, will necessarily run the business up, to the one or the other of these Operations.

The Use of the *Direct* Method, is in

{ The Investigation of *Tangents*.

{ Determining the *Maxima* and *Minima*.

{ The Invention of Points of *Inflexion* and *Retregression*.

{ Finding the *Evolutas* of Curves.

{ Finding the *Caustick* Curves.

The use of the *Inverse* Method is in

{ Rectification of Curve Lines.

{ Quadrature of Curvilinear plain Figures.

{ Universal Mensuration including the { Plaining Curve Surfaces.
Cubature of the Solids.

{ Finding Centers of Gravity.

{ Finding Centers of Oscillation.

N. B. I have included the Rectification of Curve-lines, under a Branch of the *Inverse* Me-

thod; tho' 'tis certain on the other-hand, that 'tis a consequence of the business of *Evolution*, which belongs to the *Direct* Method. But there's no doubt but 'tis most natural and just to place it where I have done. For as it stands there, 'tis the pure result of a general Process, according to the Laws of finding Fluents from Fluxions propos'd, without respect to any thing of Geometry, or to any other sort of Operation.

So that to bring it under this Branch of the Method, is to propose to find the Expression of a Curve Line, after the same vniversal way and manner, that we do all other sorts of flowing Quantities, viz. by going to work with the Fluxions given: On the other side, 'tis true, that from the Evolution of a Curve, a Right Line may be assign'd, equal in length to that Curve; but then not only the Nature of the Curve to be Rectified, but also of the Curve describ'd by the Evolution of it, must be known; and besides that the Method of Tangents must come in, or we shall never obtain the Rectification of the Curve Line that way. Upon all which accounts, 'tis much more genuine to deduce the Rectification of Curves from the *abstracted Laws of the Inverse Method*, than from the Principle of the Evolution.

I don't pretend to say, that these which I have mention'd, are all the uses that can be made of Fluxions; but only that they are those

that

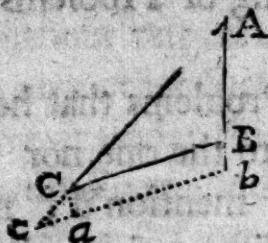
that are now a-days commonly known and discours'd of. For without doubt the vastness of the Method is not entirely shut up within these bounds. In a word, wheresoever any thing of *Motion*, *Mutation*, or *Mutable Quantities*, are concern'd, there Fluxions may be useful, in the solving of Problems relating to such Subjects.

As for the Problems that here follow, they belong neither to the one nor the other of the general Heads mention'd; nor are they entirely Applications of the *Direct*, or entirely of the *Inverse* Method, but partly of the one and partly of the other. The two first were design'd rather as farther Specimens of the way of Reasoning about Fluxions, than as any immediate Exercises of the Rules. And as the Properties of the Triangle and Circle are Fundamental to all the rest in *Linear Geometry*; so perhaps, the Properties of the Fluxions of those Figures, ought to lead the way here too, and be looked upon to be of as great use, in *Fluxional Geometry*.

Some of the rest of these Problems, are of a pretty Sublime Nature, tho' I hope render'd easie enough, and all put together, I hope may serve to entertain the young Geometer, till I (or some better hand) have the Opportunity of entertaining him farther on this Subject.

PROB. I. FIG. VII.

Any Plain Triangle being given, 'tis required to find the Proportion of the Fluxions of the Sides.



In the Triangle ACB, if the Base CB *ex. gr.* be suppos'd to move parallel to it self, then will the Sides AC, AB, CB, be augmented in the same moment of Time, into A_c , A_b , c_b , respectively. So that drawing C_a parallel to AB; the Lineolæ C_c , C_a , c_a , will be the Increments of those Sides generated in the same Moment. Now if the Line c_b be imagin'd to return back into CB, keeping still the parallel Position; the little Triangle $C_a a$, will still retain the *same Form*. But the *Form* which it now has, is that of *Similarity* to the Triangle ACB: Therefore it will retain that form of Similarity in the Moment of its vanishing. Therefore the ultimate Ratio of the Sides C_c , C_a , c_a , will be the same with that, between the Sides CA, AB, CB. But the Fluxions of the Sides CA, AB, CB, are in the ultimate Ra-
tio

tion of the *Augmenta Evanescentia*. Consequently the Fluxions of the Sides CA, AB, CB, will be as those Sides directly. Q. E. I.

C O R O L L. I.

The Fluxions of the Sides of any plain Triangle, are directly as the *Sines of the opposite Angles*. *Ex gr.* the Fluxion of CB, is to the Fluxion of AB, as the Sines of the Angle CAB, to the Sines of the Angle ACB.

Again, in a Rectangle Triangle; as *Radius* to the *Tangent* or *Secant* of either of the Acute Angles, so the Fluxion of that Side which is *Radius*, to the Fluxion of that Side which is *Tangent*, or *Secant*.

C O R O L L. II.

All the Properties demonstrated of plain Triangles with respect to any proportions of their Sides, are true also of the Fluxions of those Sides respectively.

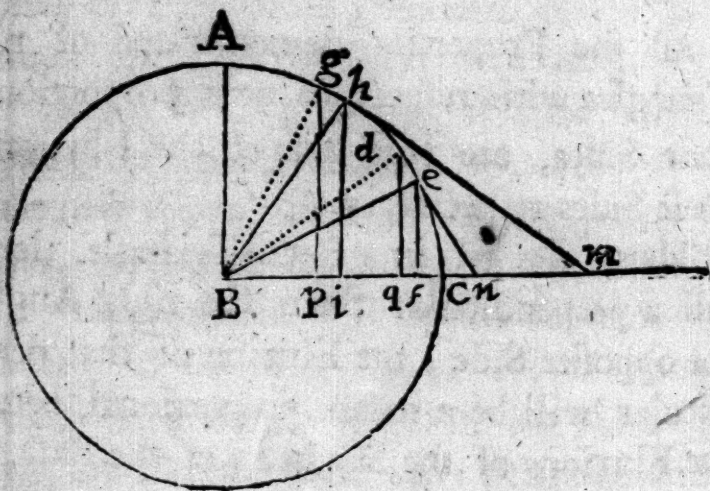
Thus, *Ex. gr.* in a Rect. Triangle, letting fall a perpendicular from the right Angle to the opposite Side; the Fluxion of that perpendicular will be a mean proportional between the Fluxions of the Segments of the Base.

C O R O L L. III.

In the *Composition and Resolution of Forces*, the Forces or Powers themselves, are directly proportional to the Fluxions of their Lines of Direction. *Ex. gr.* the Force acting with the Direction AB or AC, is to that acting with the Direction CB, as the Fluxion of AB or AC to the Fluxion of CB.

PROB. II. FIG. VIII.

Any Angles as eBC , bBC , being assign'd in a Circle; 'tis requir'd to find the proportion of the Fluxions of those Angles.



Draw

Draw the Ordinates ef , bi , and en , bm ,
Tangents at the points e , b . Put $BC=r$. Cf
 $=x$. $ef=y$. $Ci=v$. $bi=z$. Arch $Ce=c$. Arch
 $Cb=f$. If the Ordinates ef , bi , be supposed
to move in the same particle of Time into dq ,
 gp ; then fq , ip , will be the Increments of the
Abscisses, and de , gb , the Increments of the Arches
generated in the same Particle of time. At SECT.
II. 'Twas demonstrated, that the Fluxion of any
Curve-line is to the Fluxion of the Abscisse, as the
Tangent to the Subtangent. Therefore $\dot{c} : \dot{x} :: en : n f$.
also $\dot{f} : \dot{v} :: bm : mi$. But from similar Tri-
angles, $en : uf :: Be : ef$, and $bm : mi :: Bb : bi$,
Consequently in the Symbols, $\dot{c} : \dot{x} :: r : y$,
and $\dot{f} : \dot{v} :: r : z$, and $\dot{c} : \dot{f} :: \frac{r \dot{x}}{y} :: \frac{r \dot{v}}{z}$.

The Abscisse being supposed to flow uniform-
ly, then $\dot{x} = 1 = \dot{v}$, therefore $\dot{c} : \dot{f} :: \frac{1}{y} : \frac{1}{z} ::$
 $z : y$. So that the Fluxions of the Arches Ce ,
 Cb , are reciprocally as the respective Ordinates.
But (from the Elements) the Angles eBC , bBC
are measured by the Arches eC , bC on which
they stand; therefore the Fluxions of the Angles
are reciprocally as the Ordinates Q. E. I.

C O R O L L. I.

The Fluxions of the Arches diminish while
the Arches themselves increase.

C O R O L L.

C O R O L L. II.

Unequal Arches, answer to equal Intervals in the Abscisse ; and on the other hand, equal Arches to unequal Intervals.

C O R O L L. III.

At 90 gr. or in the Point *A*, the Fluxion of the Curve vanishes, or is = 0. For at the Point *A*, the Tangent and Subtangent are equal as being both infinite ; and therefore, there the Fluxions of the Curve, and of the Abscisse, are also equal ; but at *B* the Fluxion of the Abscisse vanishes ; therefore at *A* the Fluxion of the Curve vanishes also.

C O R O L L. IV.

If the Arches *Ce*, *Cb*, were in a given Ratio, *Ex.gr.* as $m:n$; that is, if $c:f::m:n$, then (by a Rule before deliver'd, the Fluxions will also be, in the same given Ratio, that is) $\dot{c}:\dot{f}::m:n$, or $\frac{r\dot{x}}{y}:\frac{r\dot{v}}{z}::m:n$, that is $nz=my$, or putting in the values of z and y , from the nature of the Circle, we have $n\sqrt{dv-vv}=m\sqrt{dx-xx}$ the Diameter of the Circle being = d . Then any one of the Abscisses, as x , *Ex.gr.* being given and esteemed as a known Quantity, the value of the other may be had, by resolving an affected Qua-

Quadratick Equation, and the help of the common Binomial Theorem. For since $n\sqrt{dv-vv} = m\sqrt{dx-xx}$, also $nn\,dv-nvvv = mm\,dx-mm\,xx$, or $nnvv-nndv = mmxx-mm\,dx$. Whence (putting $\frac{mm}{nn} = b$) $vv-dv = bxx-b\,dx$, and $vv-dv + \frac{1}{4}dd = bxx-b\,dx + \frac{1}{4}dd$, and $v - \frac{1}{2}d = \sqrt{bxx-b\,dx + \frac{1}{4}dd}$. Let this Expression be rais'd to the Power whose Index is $\frac{1}{2}$, and to that Series add $\frac{1}{2}d$, and the Sum will $= v$, an Abscisse, the Arch belonging to which, will be to the Arch (whose Abscisse is the given Quantity x) in the given Ratio of n to m .

PROB. III. FIG. IX.

If the Line AB, in any moment of Time, be supposed to be divided in extream and mean Proportion, as ex. gr in. the Point C; then the Point A continuing fixt, and the Points B and C moving in the Direction AB, 'tis requir'd to find the Proportion of the Velocities of the Points B and C; so that the flowing Line Ab, may still be divided in extream and mean Proportion, as in the Point c.

Put

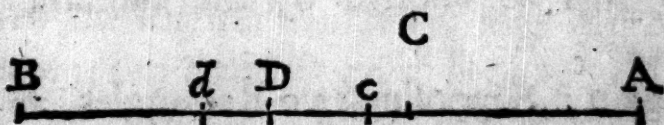


Put $AB=y$. $AC=x$. therefore $BC=y-x$.
 Let BC be the greater Segment. Because of
 the Section of the Line in extream and mean
 Proportion, 'tis $y:y-x::y-x:x$, whence yx
 $=yy-2yx+xx$, or $3yx=yy+xx$. And by
 the Direct Method, $3yx+3xy=2yy+2xx$,
 or $3yx-2xx=2yy-3xy$. From whence
 $\dot{x}:\dot{y}::2y-3x:3y-2x$; and $\dot{y}-\dot{x}:$
 $\dot{x}::y+x:2y-3x$. So that the Velocity of
 the Increment of the less Segment AC , must be
 to the Velocity of the Increment of the whole
 Line AB , as $2AB-3AC:3AB-2AC$.
 and the Velocity of the Increment of the
 greater Segment BC , must be to the Velocity
 of the Increment of the less Segment AC , as
 $AB+AC:2AB-3AC$. And the motion pro-
 ceeding after this manner, the two moving
 Points B , C . and the fixed Point A , will still
 divide the Line (in all the Augmentations or
 Diminutions of it) in extream and mean Pro-
 portion. Q. E. I.

A vast number of other Problems relating to
 the Motion of Lines and Points, which are
 directly and most naturally solv'd by Fluxions
 might

might here be propos'd to the Reader. But this Field is so large, that 'twill be besides my purpose to do any more upon this Head, than only just give some little Hints ; especially since these are of such a kind, as the Reader may pursue without much difficulty by himself. Though perhaps he may find, that the Application of Fluxions to some particular Heads of this nature, may reward him with Theorems, not altogether inconsiderable.

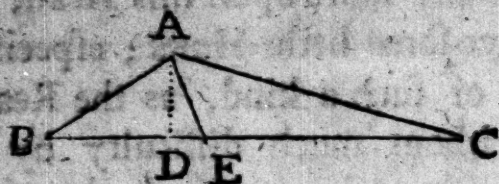
As the Line AB was at FIG. IX. suppos'd to be divided in extream and mean Proportion ; so if the Line AB (FIG. X.) be suppos'd to be



divided Harmonically (or in any other Proportion) in the Points *A*, *B*, *D*, *C* ; then to find at what rate and with what Velocities, any two, or three, or all four of these Points must move, so that the Line may still be divided by them, in the same Proportion as was at first assign'd.

Again :

Again : Supposing the Angle BAC (FIG. XI.)

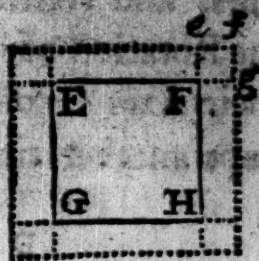
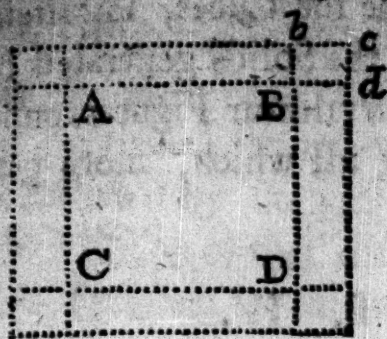


be bisected by the Line AE, as also the Lines AB and AC to revolve about the Angular Point A, and so continually to intersect the Base BC ; to find at what rate and in what proportion, the Lines AB, AC, BE, EC, must be either augmented or diminished, so that the Angle BAC may still be bisected by the constant Line AE.

Again : Supposing two Rectangles, as IM, NQ (FIG. XII. XIII.) in a given Ratio to an-



other ; as also two Squares AD, EH (FIG. XIV. XV.) in the same given Ratio ; and having



ving their Sides equal to the longer Sides of the two Rectangles respectively ; that is $AC=KM$, and $EG=NP$, then supposing these Rectangles to be augmented or diminished by a regular Flux of their Sides ; to find with what Velocities the several Sides must flow, so that the Areas of the Figures themselves may still be in the same given Ratio *ex. gr.* of m to n . If the Reader attempts the Analysis of this Problem, he will quickly see of what use the two Squares are, proportion'd in that manner as is suppos'd.

And when the Problem is solv'd with respect to the two Rectangles, 'twill be solv'd with respect to all Curvilinear Figures, that have the same Bases and Altitudes with those Rectangles, and whose Areas are in a direct constant Proportion to the Areas of those Rectangles. I say, that the same Process does also determine, at what rate such Curvilinear Figures must flow, that they may still continue to be to each other (that is, Area to Area) in a given Ratio.

The

The like may be done for Cubes, Paralleli-
pids, and consequently the Solids generated by
the Rotation of such Curvilinear Figures round
their Axes : concerning all which, more per-
haps another time.

P R O B. IV.

*The Quantity x denoting Time, and y
Velocity acquir'd in that time ; the
greatest Time, d , and the greatest Ve-
locity acquir'd in that Time, b . If
the Relation of the Velocities and
Times be thus generally express'd, viz.*

$x : d :: y : b$: 'tis requir'd to find
the proportion of the space to be describ'd
by a Motion proceeding according to
this Law, to the space describ'd by an
equable Motion, in the same Time, d ,
with a Velocity equal to the greatest
acquir'd Velocity b .

Since $x : d :: y : b$, therefore $x = \frac{d^n y^m}{b^n}$,

and by equal Extraction, $x = \frac{d y^n}{b^n}$, and by

the direct Method, $x = \frac{m}{n} \frac{d y^{\frac{m-n}{n}}}{b^{\frac{m-n}{n}}}$; Also the
Fluxion

Fluxions of the Spaces being represented by the Rectangles under the Velocities, and the Fluxions of the times; therefore the Fluxion of the space describ'd by the uniform Motion is $b \dot{x}$, and the Fluxion of that describ'd by the other, is $y \dot{x} =$ (putting in the value of $\dot{x}$ found) to

$$\frac{\frac{m}{n} dy}{b^n} ; \text{ Therefore the Spaces themselves (or}$$

the Fluents by the Inverse Method) are $b x$, and

$$\frac{\frac{mn}{mn+nn} dy}{b^{\frac{m+n}{n}}} ; \text{ that is (putting in } d \text{ for } x,$$

and b for y) $b d$, and $\frac{mn}{mn+nn} d b$. So that

the Spaces are universally as 1 to $\frac{mn}{mn+nn}$,

that is, as $mn+nn:mn$, or as $m+n:n$; that is as the Sum of the Exponents of the Times and Velocities, to the Exponent of the Velocities. Q. E. I.

From whence it appears, that if the other Motion were an uniformly accelerated one, (such as is that of Gravity) in which the Velocities are as the times, and consequently $n=m=1$; then would $mn+nn:mn::2:1$, or the Space describ'd by the equable Motion, would be double to that describ'd by the accelerate in the

N

same

same Time, and with the greatest Velocity acquir'd in that time ; which Theorem is sufficiently well known. If the Motion were such that the Velocities were in any multiplycate or submultiplycate Ratio of the Times ; as, *ex. gr.* $y:b::x^2:d^2$, or $y:b::x^{\frac{1}{3}}:d^{\frac{1}{3}}$; then $m:n::1:2$, in one case, or $m:n:1:\frac{1}{3}$, in the other, and consequently, $m+n:m::$ (that is the proportion of the spaces describ'd by the two Motions in the same time) as $3:1$, in one case, or as $\frac{4}{3}:1$, or as $4:3$, in the other. And thus interpreting the Exponents m, n , according to any Law that may be suppos'd, the proportion of the Spaces will be found in that Supposition.

C O R O L L. I.

If we conceive the Velocities y, b , as Ordinates, applied to the Abscisses x, d , which denote the respective times ; then the Figures thus form'd, shall express the proportion of the Spaces describ'd by the equable and accelerate Motion, in the same time, whatsoever the Law of that Acceleration be.

C O R O L L. II.

And these Spaces or Figures thus generated, shall be, one to another as $m+n:m$, or as $1+\frac{n}{m}:1$. So that if $\frac{n}{m}$ be the Exponent of the Series;
the

the Spaces describ'd by the two Motions, shall be as Unity increas'd by the Exponent of the Series to Unity.

C O R O L L. III.

And what these Figures are is easily determin'd from the first Hypothesis, that $x^n : d^n ::$

$y^m : b^m$. For because $x = \frac{dy^m}{b^{\frac{m}{n}}}$; therefore if $n =$

$m = 1$, then $\frac{n}{m} = 1$, and $x = \frac{dy}{b}$, or $y = \frac{bx}{d}$,

which expresses the Nature of a Triangle. But if only n or m be $= 1$, or neither n nor m be

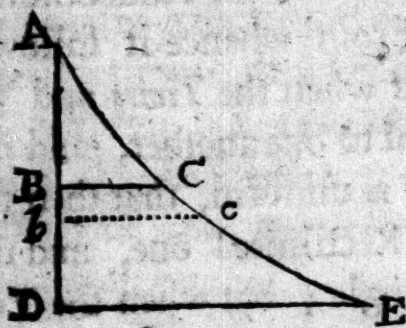
$= 1$, then the Equation $y^{\frac{m}{n}} = \frac{b^{\frac{m}{n}} x}{d}$, will ex-

press the Nature of the Paraboliform Figures in general. From whence it follows plainly enough; that when the *Times* and *Velocities* are proportional to one another; and consequently the Motion is uniform, that then the Space describ'd is a Rectilineal one, and that no other than a Triangle; but when the *Velocities* and *Times* are not thus proportion'd, but in some multiply or sub-multiply Ratio of one another, then the Figures describ'd will always be those of the Paraboliform kind, as appears from the Equation above-mention'd. For the Quantities b and d being both standing ones,

and consequently $b^{\frac{m}{n}}$ a standing Quantity, 'tis plain that $y^{\frac{m}{n}}$ is ever as x , which is the property of the Parabaliform Curves, the Index $\frac{m}{n}$ being positive.

LEMMA. FIG. XVI.

The Time that a Body spends in descending through an infinitely small Lineola, or Particle of any Line or Space; may be taken for the Fluxion of the Time, that it spends in describing that whole Line or Space.



If the Lines AB, AD, denote Times encreasing uniformly, and BC, DE, Velocities encreasing in any proportion to the Times, whatsoever; 'tis known that the Areas ABC, ADE shall express the Spaces describ'd in the times AB,

AB, AD; and so the Areas Abc , ABC , the spaces describ'd in the times Ab , AB , and consequently the Area $BbCc$ shall express the space describ'd in the time bB . Therefore if the Points C and c be infinitely near; so that the Area $BbCc$ be an infinitely small part of the Area ADE , also the Line Bb shall be infinitely small part of the Line AD . Or if $BbCc$ be as the Fluxion of the Area ADE ; that is of the space describ'd; also Bb shall be as the Fluxion of AD , that is of the time of Description. Q. E. D.

SCHOL.

The Area ADE is here expressed as a Curvilinear one, and 'tis more general to represent it as such. But in the case of the descent of heavy Bodies by the force of Gravity, that Area is a Rectilinear one, and the Figure ADE a Triangle, or the Line ACE a right Line, Line, as is known.

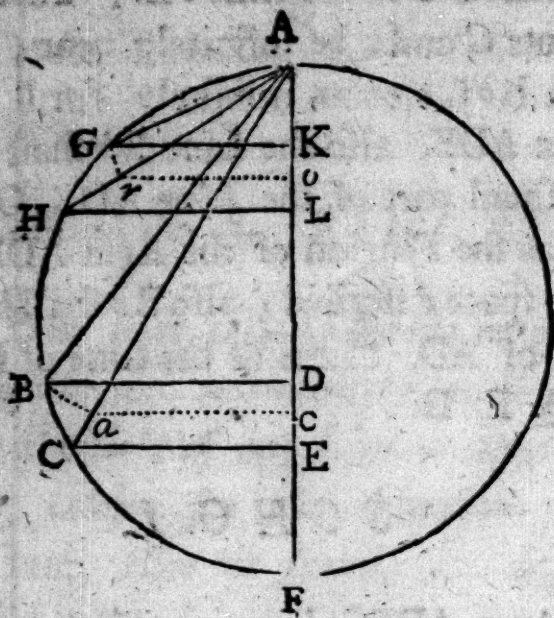
PROB. V. FIG. XVII.

The Diameter AF of the Circle ABCF, being perpendicular to the Plane of the Horizon; 'tis requir'd to find the proportion of the Times; in which a

N 3

Body

Body descending freely, by the force of its Gravity, shall describe the Chords AG, AB, taken at liberty.



From the Points *G* and *B*, draw the Ordinates *GK* and *BD*, and suppose the Chords *AG* and *AB* to move into *AH* and *AC*, in the same particle of Time ; and from the Points *H* and *C*, draw the Ordinates *LH* and *EC*. Upon the Center *A* with the Radii *AG*, *AB*, describe the small Arches *Gr*, *Ba*. Then will the *Lineola* *Hr*, *Ca*, be the Increments (of the Chords *AG* and *AB*) generated in the same particle of Time ; to which *Lineola*, the Fluxions of those Chords will be respectively proportional.

Put $AF = d$. $AD = x$. $AB = z$. Then will $z = \sqrt{dx}$, from the Nature of the Circle, and $\dot{z}$

$= \frac{1}{2} d^2 x^{-\frac{1}{2}}$, by the *Direct Method*; and consequently the Lineolæ Ca which is expounded by $\dot{x}$, will also be expounded by $\frac{1}{2} d^2 x^{-\frac{1}{2}} \dot{x}$. After the same manner, if $AK=v$. $AG=b$ shall the Lineola Hr be expounded by $\frac{1}{2} d^2 v^{-\frac{1}{2}} \dot{v}$. Now from the Principles of Mechanicks, the Times of describing the Lineolæ, Ca , Hr , shall be as those Lineolæ, *directly*, and the Velocities, with which they are describ'd, *reciprocally*. But the Velocities with which Ca and Hr are describ'd are the Velocities acquir'd by the descent thro' the Lines Aa and Ar ; for since the points r and a , are (by supposition) infinitely near to H and C ; tho' the Motion be really *accelerate*, yet the change of the Velocity in these infinitely small distances, is in a *Physical Matter*, to be neglected. So that these Velocities, are those that are gotten by the Descents thro' Aa and Ar ; that is, those gotten by the Descents through Ac and Ao ; for if ac and ro be perpendicular to AF , the Velocities gotten by the Descents thro' Aa and Ar , are equal to those gotten by the Descents thro' Ac and Ao ; as *Galileus* has demonstrated. Now the Velocities of a Body acquir'd by the Descents thro' Ac and Ao are for the reason above-mention'd, the same with those acquir'd by the Descents thro' AD and AK . Therefore the Times of describing the Lineolæ Ca Hr , are as

$\frac{Ca}{\sqrt{AD}}$, and $\frac{Hr}{\sqrt{AK}}$, or in our Symbols,
 $\frac{\frac{1}{2} d^{\frac{1}{2}} x^{-\frac{1}{2}}}{x^{\frac{1}{2}}}$, and $\frac{\frac{1}{2} d^{\frac{1}{2}} v^{-\frac{1}{2}}}{v^{\frac{1}{2}}}$, that is, $\frac{1}{2} d^{\frac{1}{2}}$

$x^{-1} \dot{x}$, and $\frac{1}{2} d^{\frac{1}{2}} v^{-1} \dot{v}$. And by the foregoing
Lemma, the Fluents of these Expressions will be as
 the Times of describing the whole Chords AC,
 AH, or (which amounts to the same thing)
 the Chords AB, AG. Now by the Rules of
 the *Inverse Method*, the Fluents of either of
 these Expressions is $\frac{d^{\frac{1}{2}}}{0}$; and so for any other

Chord whatsoever, the Expression of the Time of
 describing it must necessarily come to the same
viz. $\frac{d^{\frac{1}{2}}}{0}$. From whence 'tis certain, that the

the Times of describing all the Chords in a
 Circle are equal to one another; which was
Galileus's celebrated Theorem. Q. E. I.

S C H O L.

We have found here that the Expression of
 time for any Chord throughout the Circle
 comes out to be $\frac{d^{\frac{1}{2}}}{0}$, which expression 'tis mani-

fest, is infinite. But the Reader is not from
 thence

thence to imagine that a Body cannot descend thro' the Chord of a Circle, in less than an infinite Time, that consequence will never follow. The Business is evidently thus : the Times of the Description of the Lineolæ Hr , Ca , being universally as those little Spaces directly, and the Velocities (in r and a , that is in K and D , reciprocally ; we find from thence that the time of describing the Chord AH , comes to be as $\frac{d^{\frac{1}{2}}}{o}$, and the Time of describing the Chord

AC comes to be as $\frac{d^{\frac{1}{2}}}{o}$; from whence we infer

that these times are one to another as $\frac{d^{\frac{1}{2}}}{o}$ to

$\frac{d^{\frac{1}{2}}}{o}$, that is they are in a Ratio of Equality,

or if we should consider the matter with respect to one Chord alone ; then since the expression of the Time for any one Chord *taken at liberty*, comes to be $\frac{d^{\frac{1}{2}}}{o}$, which is a standing Quantity;

its evident from thence, that it must be the same for all Chords, and consequently that the Times of descent thro' all, are equal to one another.

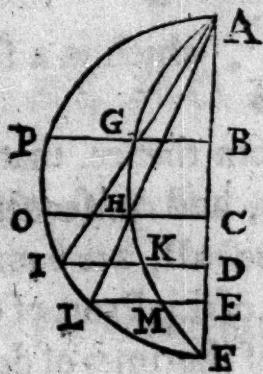
By the help of this Prop. of *Galileus* concerning the Equality of the times of descent in the Chords of a Circle ; we may demonstrate the proportion of the Times in which a Body shall descend thro' the

the Subtenses of an Ellipse; which for Varieties sake I will here do *Geometrically*.

Let AGHMF be an Ellipse, whose longer Axis is AF perpendicular to the Plane of the Horizon, and APIF a Circle whose Diameter is that longer Axis. Let any two Subtenses in the Ellipse, as AG, AH be drawn and produc'd to the Circle in I and L; as also the Ordinates GB, HC meeting the Circle in P and O. Lastly, from I and L draw the Ordinates ID, LE, cutting the Ellipse in K and M.

THEOR. FIG. XVIII.

*The Rectangle PB*LE, is to the Rectangle OC*ID, as the Square of the Time of descent thro' AG, to the Square of the Time of descent thro' AH.*



Put

Put T = Time of descent thro' AI . t = Time thro' AG . r = Time thro' AL . s = Time thro' AH . The longer Axe = a , the shorter = b .

From the Law of the descent of heavy Bodies, we have these Proportions, viz. $T : t :: \sqrt{AI} : \sqrt{AG}$, and $r : s :: \sqrt{AL} : \sqrt{AH}$. Therefore, $t : s ::$

$$\frac{T \times \sqrt{AG}}{\sqrt{AI}} : \frac{r \times \sqrt{AH}}{\sqrt{AL}}.$$

But by *Galileus's* Prop. the time through AI = Time thro' AL , that is

$$T = r; \text{ therefore } t : s :: \frac{\sqrt{AG}}{\sqrt{AI}} : \frac{\sqrt{AH}}{\sqrt{AL}}.$$

But the Triangles AGB , AID , are similar, as also the Triangles AHC , ALE ; and consequently $AG : AI :: GB : ID$, and $AH : AL :: HC : LE$;

$$\text{from whence } \sqrt{\frac{AG}{AI}} = \sqrt{\frac{GB}{ID}} \text{ and } \sqrt{\frac{AH}{AL}} = \sqrt{\frac{HC}{LE}}.$$

$$\text{Therefore, by Equality, } T : s :: \sqrt{\frac{GB}{ID}} : \sqrt{\frac{HC}{LE}}.$$

$$\text{But } \frac{GB}{ID} = \frac{GB}{KD} \times \frac{KD}{ID}, \text{ and } \frac{KD}{ID} = \frac{b}{a}, \text{ from the}$$

$$\text{Nature of the Ellipse; so that } \sqrt{\frac{GB}{ID}} = \sqrt{\frac{GB}{KD} \times \frac{b}{a}}.$$

$$\text{In like manner } \sqrt{\frac{HC}{LE}} = \sqrt{\frac{HC}{ME} \times \frac{b}{a}}.$$

Therefore dividing by $\sqrt{\frac{b}{a}}$ a standing Quantity,

$$\text{we have } t : s :: \sqrt{\frac{GB}{KD}} : \sqrt{\frac{HC}{ME}}.$$

But from
the

the Nature of the Ellipse and Circle $GB:KD::$
 $PB:ID$, and $HC:ME::OC:LE$. There-
 fore by Equality, $t:s::\sqrt{\frac{PB}{ID}}:\sqrt{\frac{OC}{LE}}$; and $t^2:$
 $s^2::\frac{PB}{ID}:\frac{OC}{LE}::PB \times LE::OC \times ID$. Q.
 E. D.

C O R O L L.

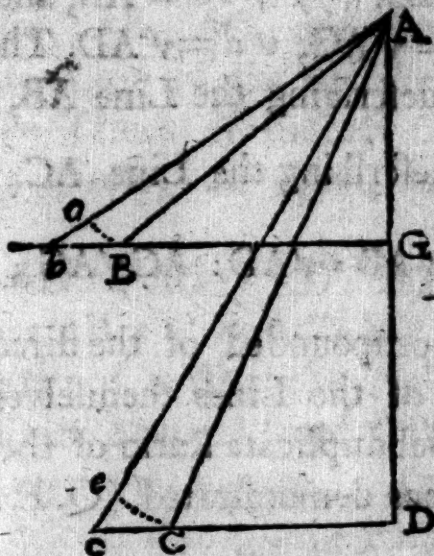
When $\overline{AB \times BF \times AE \times EF} = \overline{AC \times CF \times}$
 $\overline{AD \times DF}$; the then Times of Descent thro' the
 Elliptick Subtenses AG, AH , are equal to one
 another. For because $t^2:s^2::PB \times LE:OC \times$
 ID ; therefore when $PB \times LE = OC \times ID$ (or from
 the Nature of the Circle, when $\sqrt{AB \times BF \times}$
 $\sqrt{AE \times EF} = \sqrt{AC \times CF \times \sqrt{AD \times DF}}$, or squaring
 each Term, when $\overline{AB \times BF \times AE \times EF} = \overline{AC \times CF}$
 $\times \overline{AD \times DF}$) then will $t^2 = s^2$, and conse-
 quently $t = s$.

P R O B. VI. FIG. XIX.

*The Line AGD being Perpendicular to
 to the Horizontal Lines, BG, CD,
 'tis requir'd to find the Proportion of
 the Times, in which a Body shall de-
 scribe the Lines AB, AC, which are
 scribe*

neither of equal Altitudes, Lengths,
nor Inclinations.

Let the Lines AC, AB move into Ac, Ab,
in the same Moment.



Put $AG = b$. $AD = d$. $BG = y$. $CD = z$;
 $AB = f$. $AC = v$. Then $Bb = j$. $Cc = i$. ab

$$= j = \frac{y\dot{y}}{\sqrt{bb + yy}}. \quad ec = i = \frac{z\dot{z}}{\sqrt{dd + zz}}. \quad \text{Then}$$

the Velocities acquir'd in the Points a and e ,
being taken for those acquir'd by the descents
through the Lines AG, and AD, which Velo-
cities are as $\sqrt{b}$, and $\sqrt{d}$; the times of descri-

bing the Lineolæ ba , ce , will be as $\frac{y\dot{y}}{\sqrt{bb + yy} \times \sqrt{b}}$

and $\frac{z\dot{z}}{\sqrt{dd + zz} \times \sqrt{d}}$. The flowing Quanti-

ties of which Expressions (by the Rules of the
Inverse

Inverse Method already delivered) are $\frac{\sqrt{yy+bb}}{\sqrt{b}}$, and $\frac{\sqrt{zz+dd}}{\sqrt{d}}$; and these (by the Lemma) are as the Times of describing, the Lines Ab , Ac , or (which is all one) the Lines AB , AC . But $\sqrt{yy+bb} = AB$, and $\sqrt{zz+dd} = AC$, $\sqrt{b} = \sqrt{AG}$, $\sqrt{d} = \sqrt{AD}$. Therefore the Times of describing the Line AB , is to the Time of describing the Line AC , as $\frac{AB}{\sqrt{AG}}$: $\frac{AC}{\sqrt{AD}}$, or as $AB \times \sqrt{AD} : AC \sqrt{AG}$; that is, in the Ratio compounded of the direct Ratio of the lengths of the Lines themselves, and the reciprocal Subduplicate Ratio of the Altitudes; as *Galileus* has demonstrated. Q. E. I.

C O R O L L.

From this general Proposition, the rest relating to this matter may be inferr'd. For,

1. If $AG = AD$, that is, if the Altitudes of the Lines describ'd are equal, then the Times of Description will be as the lengths of those Lines, *directly*.

2. If $AB = AC$, or the Lines describ'd be equal, then the Times of Description, are in the Subduplicate Ratio of the Altitudes *reciprocally*. And these two Theorems *Galileus* demonstrates severally.

3. If

3. If $AB : AC :: \sqrt{AG} : \sqrt{AD}$, then $AB \times \sqrt{AD} = AC \times \sqrt{AG}$, and so the Times of describing those Lines AB , AC , are equal. Now this Property agrees to a Circle (*ex. gr.* whose Diameter lies in AD , and whose Periphery passes through the Points, A, b, c , or A, B, C) that the Subtenses, AB , AC , are directly in the Subduplicate Ratio of the Abscisses AG , AD ; therefore the Times of descent through the Chords of a Circle, are equal, as was found before.

4. If $\sqrt{AD} : \sqrt{AG} :: CD : BG$, then the Times are as $AB \times CD : AC \times BG$. Now this is the Property of a Parabola (whose Vertex *ex. gr.* is A , its Axe AD , and Ordinates BG , CD) as is very well known. Therefore the Times of descent through the Parabolick Subtenses, are as those Subtenses, *directly*, and the correspondent Ordinates, *reciprocally*; as *Torricellius* has demonstrated, Prop. 27. Lib. 1. *De Motu Gravium Descendentium*.

I shall only say farther, that as these are the principal Theorems of that noble Author (*Galileus* I mean) his excellent third Mechanick Dialogue; so I believe there are very few (if any of them at all) which may not be thus investigated: Which therefore I shall now leave to the young Geometer's own diligent Enquiry.

P R O B.

P R O B. VII.

To find the Law of the Acceleration, in the description of Curvilinear Figures.

In the first Section we shew'd, that the Generation of any sort of Curve might be accounted for from the compound Motion of a Point. And that considering one of the two compounding Motions, as equable and uniform, the other would be an accelerated one; the Condition and Law of which Acceleration would be different, according to the Nature of the Curve so generated. If then we make the Motion in the direction of the Abscisse, *Ex. gr.* (of any Curve) to be accelerate, and consequently that in the direction of the Ordinate to be uniform; then as the Quantity that expounds the Fluxion of the Abscisse, is to the Quantity that expounds the Fluxion of the Ordinate, so is the Velocity of the accelerate Motion in the Direction of the Abscisse, to the Velocity of that uniform in the Direction of the Ordinate, for that particular Point in the Curve. And so likewise, as the Quantity that expounds the Fluxion of the Curve Line to the Quantity that expounds the Fluxion of the Ordinate or Abscisse, so the Velocity of the compound Motion in any Point, to the Velocity

locity of either of the compounding Motions, in the direction of the Ordinate or Abscisse. But from the Property of the Curve given, the Relation of the Fluxions of the Curve, Ordinate and Abscisse may be had in *finite Terms*. And consequently the proportion and relation of these Velocities, for any point in the Curve Q. E. I.

Ex. gr. Suppose the Nature of a Curve express'd by this Equation, $x^3 - x^2 y = a y y$. Then

by the Direct Method, $3x^2 \dot{x} - 2y \dot{x} x = 2a y \dot{y}$

$+ x \dot{y}$, from whence $x : y :: 2a y + x : 3x^2 - 2yx$,

and squaring $x : y :: \frac{2a y + x^2}{3x^2 - 2yx}$,

and compounding, $\frac{x^2 + y^2}{3x^2 - 2yx} :: \frac{2a y + x^2}{3x^2 - 2yx}$

$+ 3x^2 - 2yx : 3x^2 - 2yx$, and taking the

square Roots; $\sqrt{x^2 + y^2} : y :: \sqrt{2a y + x^2} +$

$3x^2 - 2yx : 3x^2 - 2yx$. So that the Abscisse

being x , and the Ordinate y , the Velocity of

the compound Motion in any point of the

Curve, shall be to that in the direction of the

Ordinate, for the same point, as the former of

these Expressions to the latter; for $\sqrt{x^2 + y^2} : y :: \sqrt{2a y + x^2} +$

$3x^2 - 2yx : 3x^2 - 2yx$. So that the Abscisse

being x , and the Ordinate y , the Velocity of

the compound Motion in any point of the

Curve, shall be to that in the direction of the

O

COROLL.

C O R O L L. I.

Hence these Velocities may be expounded also by the Tangent, Ordinate and Subtangent: For we have demonstrated, that these are proportional to the Fluxion of the Curve, of the Ordinate and of the Abscisse.

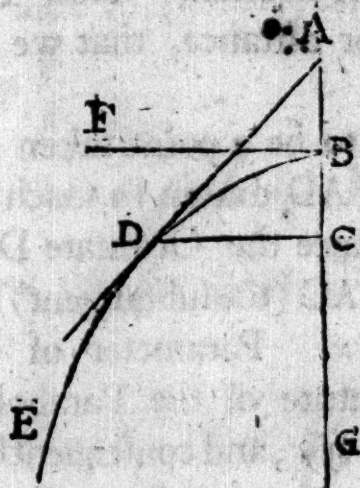
C O R O L L. II.

Hence also those noble Theorems that determine the proportions of the Velocities, in the compound Motions of Projects (or Bodies otherwise describing Curve Lines) may without much difficulty be discover'd. For an Example of which, we will take that most universal and Ingenious Problem of *Galileus*, which *Toricellius* solves *Prop. 17. Lib. 1.* about finding a Point above the Vertex of a *Parabola* given; from whence if an heavy Body falls, and be afterwards impell'd Horizontally with a Velocity equal to that acquir'd by the descent, it shall describe the same *Parabola*.

P R O B. F I G. XX.

The Parabola BDE being given; 'tis requir'd to find a Point beyond B the Vertex of the Curve: So that a Body impell'd in the Horizontal Direction,

rection, or vertical Tangent BF, with a Velocity equal to that gotten by the Descent from that Point to the Point B; may (by the concurrence of its Gravity with this equable Force) describe the same Parabola.



Now in order to this, let it be consider'd first of all, that as a Parabola is describ'd by the Composition of two Motions, an equable and an uniformly accelerated one; so according to the different force or greatness of that equable Motion, a Parabola of different bigness and parameter is describ'd. Now the Body in the accelerate Motion of perpendicular Descent, acquiring continually greater degrees of Velocity, it being suppos'd at perfect rest in B the Vertex of the Parabola, shall in some Point

or other of the Axe BG, have acquir'd such a degree of Velocity, as is equal to that Velocity of the uniform Motion before-mention'd. And consequently taking a point above as far distant from the Vertex, as this point is below it; the Body by falling from that point above, to the Vertex of the Parabola, shall acquire such a degree of Velocity as is equal to that of the uniform Motion. Now 'tis this particular length or distance, that we propose here to find out.

Suppose A to be a point taken at liberty in the Axis, and AD drawn to touch the Curve in D ; from whence the Ordinate DC . Put $DC = y$. $BC = x$. AC (the subtangent) $= t$. Therefore $AB = t - x$. Parameter of the Axe $= p$. From the nature of the Parabola, $px = yy$, whence $px = 2y^2$, and consequently $x : y :: 2y : p$. That is (by what was shewn in the preceding Problem) the Velocity of the Accelerate Motion in the direction of the Axe, at the Point C , is to the Velocity of the equable Motion, in the direction of the Ordinate, as $2y : p$. Therefore when these Velocities are equal; that is, when $x = y$, then $2y = p$ or $y = \frac{p}{2}$, or $yy = \frac{pp}{4}$, or $px = \frac{pp}{4}$, or $x = \frac{p}{4}$. And since $t = 2x$, (from the Nature of the Curve) therefore also $t - x$ or the distance above the Vertex, shall $= \frac{p}{4}$. So that in order

to satisfy the demand of the Problem, the Line AB must be equal to $\frac{1}{4}$ of the Parameter.

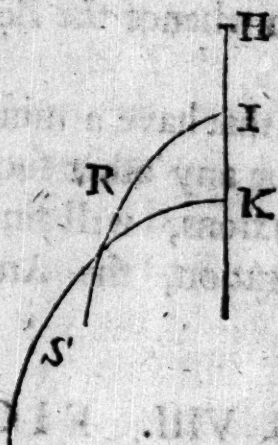
Q. E. I.

S C H O L.

We may come to the same Conclusion thus also. Because $x : y :: t : y$, therefore when $x = y$, t must $= y$, and t being $= 2x$, then x must $= \frac{1}{2}y$, and so $4xx = yy = px$, and consequently $x = \frac{p}{4}$.

C O R O L L. I.

'Tis evident that the point C is the Focus of the Parabola.



C O R O L L. II.

In different Parabolas (FIG. XXI) IR, KS, describ'd after this manner, whose Vertices are I and K ; if the Body falls from the
O 3
same

same point H; the Velocities of the equable or Horizontal Motions, will be in the Subduplicate Ratio of the Parameters, directly.

And they would be in the same proportion, if the Vertices of two Parabollas, NO, NP were co-incident (as at N, FIG. 22) and so



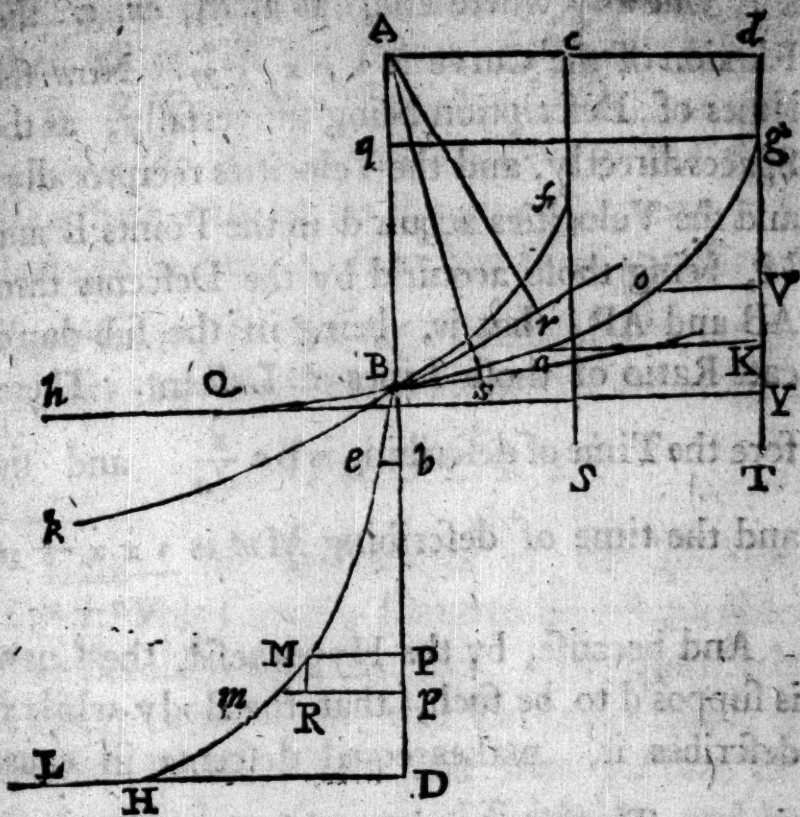
the Points from whence the Body fall, were L, and M.

N. B. Those that have a mind to try more of Torricellius's, or any other such Props. by the Method of Fluxions, will find by a just and genuine Application, the Analysis of them not difficult.

PROB. VIII. FIG. XXIII.

'Tis requir'd to find the Nature of the Curve, in which a heavy Body shall descend, by the Force of its Gravity, without any Acceleration.

Let



Let ABD be a right Line perpendicular to the Plane of the Horizon ; and the Curve BMH be the Curve required, whose Axe BD is in the same strait Line with AB. The Body now is suppos'd to fall from A to B, and from thence to commence a Motion in the Curve BMH ; in which Curve it must still move with the Velocity acquir'd by the descent from A to B. Put $AB=a$. $BP=x$. $PM=y$. $Pp=\dot{x}$ Bb . $Rm=\dot{y}$. In the Point B, the Fluxion of the Ordinate vanishes, as is very evident ; therefore there the Fluxion of the Curve (which universally express'd, is $\sqrt{\dot{x}^2 + \dot{y}^2}$) becomes

$\dot{x}$. But any where else, as at M , *ex. gr.* the Fluxion of the Curve is $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. Now the times of Description being universally, as the Spaces directly, and the Velocities reciprocally; and the Velocities acquir'd in the Points B and M , being those acquir'd by the Descents thro' AB and AP ; that is, being in the sub-duplicate Ratio of those Lines of Descent. There-

fore the Time of describing is $B e \frac{\dot{x}}{\sqrt{a}}$: and the and the time of describing $M m$ is $\frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\sqrt{a+x}}$.

And because, by the Hypothesis, the Curve is suppos'd to be such, that the Body while it describes it, makes equal descents in equal times. Therefore we have this Equation $\frac{\dot{x}}{\sqrt{a}} =$

$\frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\sqrt{a+x}}$; from whence (squaring) $a\dot{x}\dot{x} +$

$x\dot{x}\dot{x} = a\dot{x}\dot{x} + a\dot{y}\dot{y}$, and $x\dot{x}\dot{x} = a\dot{y}\dot{y}$, and

(extracting the square Root) $x^{\frac{1}{2}}\dot{x} = a^{\frac{1}{2}}\dot{y}$, and

by the Inverse Method $\frac{2}{3}x^{\frac{3}{2}} = a^{\frac{1}{2}}y$, and (squa-

ring) $\frac{4}{9}x^{\frac{3}{2}} = ayy$, or $x = \frac{2}{4}ayy$, which is the

Equation of the Curve sought; and shews that

it is a semicubical Paraboloid in which x or BP

is an Ordinate, and y or PM an Abscisse, conse-

quently the Vertex B , and vertical Tangent BP ,

BP, and whose Parameter is $\frac{2}{4} a$, or $\frac{2}{4} AB$; that AB is $\frac{4}{9}$ of the Parameter. Q. E. I.

N. B. From the Hypothesis, the Quantities x and y do vanish both together; and therefore there can be no standing Quantity to be brought in here, to compleat the Expression of the Fluent; so that the Fluents of $x^{\frac{1}{2}} \dot{x}$, and $a^{\frac{1}{2}} \dot{y}$, are only those Expressions that are assign'd.

C O R O L L. I.

Hence, the Rectification of the *Isochronal Curve*. This Curve is found to be a semicubical Paraboloid, and 'tis true, that from other Principles we know, that Curve is capable of a perfect Rectification. But then in that case, 'tis rectified as a *semicubical Paraboloid*, and here I speak of the Rectification of it, as the *Isochronal Curve*; or as a consequence of this supposition, that equal spaces of descent, are made in equal times; which way of proceeding is vastly different from the other, upon all accounts. Now, I say, this follows; for since

by the Hypothesis, $\frac{\dot{x}}{\sqrt{a}} = \frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\sqrt{a+x}}$, there-

fore putting the length of the Curve Line BM = c , then $Mm = \dot{c} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, therefore

$\frac{\dot{x}}{\sqrt{a}} = \frac{\dot{c}}{\sqrt{a+x}}$, and $\dot{x} \sqrt{a+x} = \sqrt{a} \times \dot{c}$. Now

the

the Fluent of the Expression $\sqrt{a+x}$, is
(by the Inverse Method) $\frac{2}{3} \sqrt{a+x}^{\frac{3}{2}} = \frac{2a + 2x}{3}$

$\sqrt{a+x}$; for the Fluxionary Expression without the *Vinculum*, is the Fluxion of the Quantity contain'd under the *Vinculum*. Therefore it

follows (since $\sqrt{a} \times c$, is the Fluent of $\sqrt{a} \dot{c}$)

that $\sqrt{a} \times c = \frac{2a + 2x}{3} \sqrt{a+x}$, and consequently

$$\frac{\frac{2a + 2x}{3} \sqrt{a+x}}{\sqrt{a}} = c = \text{the length of the Curve.}$$

Line BM.

Schol. From the Nature of the Curve, we

$$\text{have } a = \frac{4}{9} \frac{x^3}{y^2}, \text{ so that } c = \frac{\frac{8x^3}{yy} + 8x\sqrt{\frac{4x^3}{9y^2} + x}}{27 \frac{x^{\frac{3}{2}}}{y}}$$

And consequently if we put $x=0$, then c will $=0$ also, that is the Curve Line and the Abscisse vanish together. Which I note, to prevent a wrong Conclusion, that might possibly have been drawn from the Formula, $c =$

$$\frac{\frac{2a + 2x}{3} \sqrt{a+x}}{\sqrt{a}}; \text{ where putting } x=0, \text{ there}$$

is a standing Quantity left, and so the Curve
Line

Line might have been suppos'd of some Magnitude, when the Abscisse was made to vanish. But, I say, the Quantity a involves x in the Expression of it, from the Nature of the Curve; and therefore x being in every Term of the Expression that gives the length of the Curve Line when x vanishes, that whole Expression vanishes also.

C O R O L L. II.

The Times spent in describing any Portions of the Curve, as BeM , BeH ; are as the correspondent Abscisses BP , BD , directly. For in what times those Portions of the Curve are describ'd in the same times the respective Abscisses are describ'd by the Motion of the perpendicular descent. But this descensive Motion is an equable one, and consequently BP and BD are as the Times in which they are describ'd; and therefore as the times of describing the Portions BeM , BeH . This follows also immediately from the Calculus. For the

time of describing the particle Mm , is $\frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\sqrt{a - x}}$

which is $= \frac{\dot{x}}{\sqrt{a}}$. Therefore (by the Lemma to PROB. V.) the Fluent of $\frac{\dot{x}}{\sqrt{a}}$, shall be the time of describing any Portion of the Curve BeM , *ex. gr.* But the Fluent of $\frac{\dot{x}}{\sqrt{a}}$ is $\frac{x}{\sqrt{a}}$ and

and consequently a being constant, the time is as the Abscisse $x = BP$.

C O R O L L. III.

If BD were $= 2AB$, then the descent thro' BD would be made in the Curve, in the same time, that the Body fell from A to B . For, the descensive Motion in the Curve is uniform, and perform'd with the same Velocity as is gotten by the accelerate Motion in the descent from A to B . Therefore BD being equal to $2AB$, will be describ'd in the same time as AB ; by a known Prop. of *Galileus*, and as we have also shew'd at PROB. IX.

C O R O L L. IV.

The Line Acd being parallel to the Horizon, and the Lines cf , dg , &c. perpendicular to the same; 'tis plain that an infinite number of Isochronal Curves, as Bk , Bb , &c. may be drawn through the Point B ; which are all of them Semicubical Paraboloids, and, consequently, all of them Curves of the same Species, and differing only in the Magnitude of their Parameters. And consequently, that a Body at B , with the Velocity, acquir'd by the descent from the Horizontal Line Acd , may descend through any one of this infinite number of Isochronal Curves. Neither is it any matter which

which of the number be fix'd upon, since either of them all will serve, provided it be plac'd so, that cf , or dg , the distance of the Vertex of the Curve, from the Horizontal Line, be $\frac{4}{9}$ of the Parameter of the Curve fBk , or gBb . Farther; 'tis all one, whether the heavy Body (which is to descend *Isochronically*, *ex. gr.* in the Curve Bk) comes to the Point B , by the way cfB , or by any other way, as dgB , or AB ; since in either case, it will describe the Curve with a certain determinate degree of Velocity, and so shall descend equably thereupon. 'Tis true, if the Body be to move in the Curve Bk , and comes to B by the way AB , then it comes with a Velocity equal to that acquir'd by the descent through AB ; if it comes by the way cfB , or dgB , then in the former case it comes with no more Velocity than that acquir'd by the descent thro' cf , and in the latter with no more than that acquir'd by the descent through dg : And so the Velocities with which the same Curve will be describ'd, will be different, according to the different Paths the Body takes to come at it. But notwithstanding in either case, the Body will descend *equably* and *without any Acceleration*, for that properly results immediately from the nature of these Curves, and will take place, whatever Collateral Circumstances of the Motion, fall in to be consider'd. 'Twould be the same thing, if we should suppose

pose the Body at B, not to have acquir'd any Velocity, by falling from any height; but to be some way or other impress'd by some force, which should amount to as much, as such a degree of Celerity, impressed in such a Direction. For in this Case also, it would be true, that the Body might descend through any one of all those Isochronal Curves. Thus, if the Line Br were a Tangent to the Curve fBk at the Point B, and the Body were to move in that Curve; then if it be impress'd with an Impetus at the Point B, equivalent to what is gotten by the descent through cf , and in the direction of that Tangent; it shall describe the Paraboloid Bk , all one, as it would have done if it had mov'd to the Point B through the Path cfB . Or if in that same Direction of the Tangent, it were impressed with an Impetus, equivalent to that gotten by the descent through AB , it shall describe the Curve in the same manner, as if it had fallen from Acd by the Line AB .

C O R O L L. V.

Of all the Isochronal Curves, intersecting each other in the Point B, that Curve gives the Body the *swiftest descent*, whose Vertex is that Point B, and consequently AB its Vertical Tangent; that is, the Curve BMH is here the Curve of *swiftest (equable) descent*.

This

This easily follows; whether we suppose the Body to describe the whole Curves fBk , gBb , beginning at the Points c and d ; or only the parts of them, *viz.* Bk , Bb , beginning its motion at the Point A .

1. If the Body falls from A , it describes the Curve BMH with a Velocity acquired by the descent through AB . If it falls from c , it describes the Curve fBk with the Velocity acquir'd by the descent through cf . But $AB > cf$; and therefore the Velocity in the Curve BMH , is greater than the Velocity in the Curve fBk ; and therefore the time of descending any given Space, is less in that Curve, than in the other; and therefore the Body moves swifter in that Curve than in the other. After the same manner we may prove the Body to move swifter in that Curve BMH , than in any other whatsoever, whose Vertex is in a Line (parallel to AB) drawn to any Point between A and c ; and much more, that 'tis swifter, than in any other Curve as gBb , whose Vertex is in a Line beyond the Point c .

2. Suppose the Body were to fall from A , and so to describe each of the Curves, Bb , Bk , BM , successively. Let the Lines Br , Bs , touch the Curves fBk , gBb , at the Point B , and Ar , As Perpendiculars to those Tangents. Now the Body descending in the Direction AB , gives a greater stroke upon the Curve gBb , than
upon

upon the Curve fBk , and consequently looses more of its Motion, by falling upon the former than upon the latter Curve. So that the descent will be slower (though uniform) in the Curve gBb , than in fBk ; and swiftest of all in the Curve BMH , whose Vertical Tangent AB , is coincident with the *Line of Direction* of the Body's Motion, and where consequently the *Shock* against the Curve is of all the least possible.

C O R O L L. VI.

If the Lines BD , fS , gT , are equal; the Times wherein the Body shall have descended those lengths, as it describes the Curves BMH , fBk , gBb , supposing it to fall from the Points A , c , d ; will be in the subduplicate Ratio of the Parameters, reciprocally. For the Spaces by the Hypothesis being equal, and the Motions uniform, then by the Principles of *Mechanicks*, the Times shall be reciprocally as the Velocities. But the Velocities with which the Body moves in these Curves, are those acquir'd by the Descents thro' AB , cf and dg , which Velocities are to each other in the subduplicate Ratio of the Lines AB , cf , dg ; which Lines AB , cf , dg , are each $\frac{1}{2}$ of the respective Parameters of those Curves. Therefore, &c.

C O R O L L.

C O R O L L. VII.

If the Lines BP , gv , are describ'd in equal times, they are directly in the subduplicate Ratio of the Parameters of those Curves in which the Body has thus descended. For the Descent being uniform, and times equal (by the Hypothesis) the spaces describ'd shall be directly as the Velocities; that is, directly in the subduplicate Ratio of AB , cf , dg , which are as the Parameters respectively.

Hence the flowing Problem is easily solv'd.

P R O B.

A Body moving in one Isochronal Curve, as BMH , and at any instant of Time, its place in that Curve being assign'd; 'tis requir'd to find at what Point in another Isochronal Curve as gBh a Body (describing the same) shall be at the same Moment.

Here the distances of the Points, B , g , from the Horizontal Line Ad , are suppos'd to be given; and consequently the Parameters of those Curves to be known. The Bodies also are suppos'd to fall from the Horizontal Line Ad ;

P

or

or to begin their Motion from thence, as was before all along suppos'd.

C A S E. I.

Let the Bodies be imagin'd to arrive at the Points B and g, the Vertices of the two Curves, in the same instant.

Then 'tis certain, that the Body at A must set out from thence, before the Body at d sets out from thence. For I suppose them to move only by the force of their Gravity: Now by the Law of the Descent of heavy Bodies, if the two Bodies should set out together from the Points A and d, the Body at d will reach the Point g much sooner, than the Body at A will arrive at the Point B, and the former time will be to the latter, as $\sqrt{dg} : \sqrt{AB}$, or in the subduplicate Ratio of the Spaces run.

This premis'd, I say, that supposing one Body to be at M, in the Curve BMH, and from thence drawing the Ordinate MP, if in the Line gT the Axis of the Curve gBb, we set off $gV = \frac{BP \times \sqrt{dg}}{\sqrt{AB}}$, and from thence draw the Ordinate Vo, it shall cut the Curve in the point sought; that is, the one Body shall be at the Point o in the same Moment that the other is in the Point M. The Demonstration of which is easie. For because (by the Hypothesis) the Bodies arrive at the Points B and g, in the

the same Moment; therefore they will commence the Motions of Equable descent in the same Moment of Time; each in its respective Curve. Now then in the same time that the Body *A* has descended the length *BP* in the Curve *BMH*, the Body *d* in the Curve *gBb* shall have descended a length which shall be to *BP*, as the Velocity in the Curve *gBb*, to the Velocity in the Curve *BMH*; for these Motions of descent are uniform. But the Velocities in these Curves (by the foregoing *Coroll.* are directly in the Subduplicate Ratio of the Parameters, that is, as $\sqrt{dg}$ to $\sqrt{AB}$. Therefore when the descent *BP* is made in the Curve *BMH*, the descent $gV = \frac{BP \times \sqrt{dg}}{\sqrt{AB}}$ shall be made in the Curve *gBb*. Q. E. D.

C A S E. II.

Let the Bodies set out from the Points A and d, at the same instant of Time.

Then 'tis plain that the Body which falls from from *d* will arrive at *g* the Vertex of the Curve *gBb*, before the Body from *A* will reach *B* the Vertex of the Curve *BMH*. Consequently the Body from *d*, will have begun the equable Motion of descent, in the Curve *gBb*, while the Body from *A* is yet moving accelerately in the Line *AB*. Now the Body that fell from *A*, being in any Moment of time suppos'd to be (as before) in the point *M* of the Curve

P 2
BMH

BMH, the Point $\mathcal{Q}$, in the Curve gBb , where the Body that fell from d , shall be at the same instant, is thus determin'd. From the Point g , set off $gK = 2\sqrt{AB \times \sqrt{dg}} - 2dg$; and from the Point K , set off $KY = \frac{BP \times \sqrt{dg}}{\sqrt{AB}}$, and thro' the Point T draw the Ordinate YQ , cutting the Curve in $\mathcal{Q}$, which is the Point sought. The Demonstration of which is as follows; and as I have (for the more easie apprehension of the Matter) divided the construction in two parts, so for the same Reason, I shall likewise do the Demonstration.

First, therefore, I say, that when the Body from A has descended the length AB , the Body from d has descended the length dK , or $dg + gK$. which Line gK is of that value that was assign'd above.

For since (by the Hypothesis) the Bodies set out from the Points A and d at the same instant; therefore at what the Body d is in g , the Body A shall be in q ; the Line gq being parallel to Ad , and so $Aq = dg$. So that it remains to be demonstrated, that what time the Body A spends in descending the length qB by the uniformly accelerate Motion, the same time does the Body d spend in descending the length gK , by the equable Motion, with the Velocity acquir'd by the descent thro' dg . Now in order to this, let us consider what space the Body

A

A would descend by the equable Motion in the Curve *BMH*, and with the Velocity acquir'd by the Descent thro' *AB*, in the same time that it falls the space *qB* by the uniformly accelerate Motion. For when this is found, we may then find by proportion, what space the Body *d* shall descend, by the equable Motion in the Curve *gBb*, and with the Velocity acquir'd by the descent thro' *dg*, in the same time that the Body *A* falls the space *qB* by the accelerate Motion. Now 'tis a known Theorem, that in the same time the Body falls from *A* to *B*, or describes the Line *AB* by the uniformly accelerate Motion; it shall in an equable Motion with a Velocity equal to that gotten by the descent from *A* to *B*, describe $2AB$. So again, in what time the Body describes the Line *Aq* by the uniformly accelerate Motion; in the same time in an equable Motion with a Velocity equal to that gotten by the descent thro' *Aq* it shall describe $2Aq$. And therefore in an equable Motion, with a Velocity equal to that gotten by the descent thro' *AB*, it shall describe $\frac{2Aq \times \sqrt{AB}}{\sqrt{Aq}}$ (or $2 \sqrt{AB} \times Aq$) in the same time, that it describes *Aq* by the uniformly accelerate Motion. For in those two equable Motions, the Spaces run in the same time, are as the Velocities, which Velocities are those acquir'd by the descents through

P 3 *Aq*

Aq and AB , and are therefore (by the Law of the descent of heavy Bodies) as $\sqrt{Aq}$ to $\sqrt{AB}$.

From hence then we come to this Conclusion : Since in the same time, that AB is described by the uniformly accelerate Motion, the Space $2AB$ is describ'd in the equable Descent (in the Curve BMH) with the Velocity equal to that gotten by the descent through AB . And also, since in the same time that Aq is describ'd by the uniformly accelerate Motion, the space $2\sqrt{AB \times Aq}$ is describ'd in the equable Descent (in the Curve BMH) with the Velocity equal to that gotten by the descent thro' AB . Therefore in the same time that $AB - Aq$, or qB is describ'd by the uniformly accelerate Motion; the Space $2AB - 2\sqrt{AB \times Aq}$, shall be describ'd in the equable descent (in the Curve BMH) with the same Velocity as before, viz. that which is acquir'd by the descent thro' AB . For qB is the difference between AB and Aq , and the time of describing qB is the difference of the times of describing AB and Aq . And therefore whatever the Spaces be, that are describ'd by the *equable* Motion, in the same times that AB and Aq are describ'd by the uniformly Accelerate; the difference of those Spaces, is apparently the Space that is describ'd by the equable Motion, in the difference of the Times that AB and Aq are describ'd by the *Accelerat* . And

And from hence I inferr next : That since the Velocity of the equable Descent in the Curve BMH, is to the Velocity of the equable Descent in the Curve qBb , as $\sqrt{AB}$ to $\sqrt{dg}$;

and since the space $2AB - 2\sqrt{AB} \times Aq$, is describ'd, by the equable Descent in the Curve BMH (with a Velocity that is as $\sqrt{AB}$) in the same time that qB is describ'd by the uniformly accelerate Motion. Therefore by proportion,

$$\sqrt{AB} : \sqrt{dg} :: 2AB - 2\sqrt{AB} \times Aq : \text{to a 4th,}$$

$$\text{which is} = \frac{2AB \times \sqrt{dg} - 2\sqrt{AB} \times Aq \times \sqrt{dg}}{\sqrt{AB}},$$

which is $= 2\sqrt{AB} \times dg - 2dg$ (because $dg = Aq$. And this therefore is the space that the Body d will describe by the equable descent in the Curve gBb , in the same time, that the Body A describes the Line qB , by the uniformly accelerate Motion. From hence then it appears, that when the Body A is just come to the Point B , and there beginning the equable Motion in the Curve BMH; the Body d , has already advanc'd so far in its equable Motion in the Curve gBb , as to have describ'd the length $2\sqrt{AB} \times dg - 2dg$, in the Axis below the Point g . And therefore, I say, that if we set off gK equal to this value, and draw Ka an Ordinate to that Point, cutting the Curve in a ; that the Body d shall be in the Point a , at the same instant that

the Body *A* is in the Point *B*. Which was the first thing to be demonstrated.

The second thing to be demonstrated is, that setting off $KY = \frac{BP \times \sqrt{dg}}{\sqrt{AB}}$; when the Body *A* in the Curve *BMH*, has made the descent *BP*, that then the Body *d*, in the Curve *gBh* has made the descent $KY = \frac{BP \times \sqrt{dg}}{\sqrt{AB}}$. And this is done without the least difficulty. For the Motions of descent in these Curves being equable, the spaces run in the same time will be as the Velocities, that is as $\sqrt{AB}$ to $\sqrt{dg}$. So that when the Body *A* has descended the length *BP* from *B*, or is in the Point *M*, in the Curve; at the same time the Body *d* shall have descended the length $\frac{BP \times \sqrt{dg}}{\sqrt{AB}}$ from *K*, that is, drawing the Ordinate *YQ*, shall be in the Point *Q* in the Curve *gBh*. Q. E. D.

Therefore now to satisfy the demands of the Problem, as to this 2d (and more difficult) Case. If the Bodies *A* and *d* set out from the Points *A* and *d* together; and at any instant reckoning from that first Moment of their Motion, the descent that *A* has made from the Horizontal Line be given, as, *ex. gr.* *AB + BP* and consequently its Place *M*, in the Curve *BMH*; then setting off from *d* in the Horizontal Line, the length *dY* ($= dg + gK + KY$)
 $= dg$

$$= dg + 2\sqrt{AB} \times dg - 2dg + \frac{PB \times \sqrt{dg}}{\sqrt{AB}} = 2$$

$$\sqrt{AB} \times dg - dg + \frac{BP \times \sqrt{dg}}{\sqrt{AB}}; \text{ the Ordinate}$$

YQ shall intersect the Curve in Q the Point desir'd. Q. E. I.

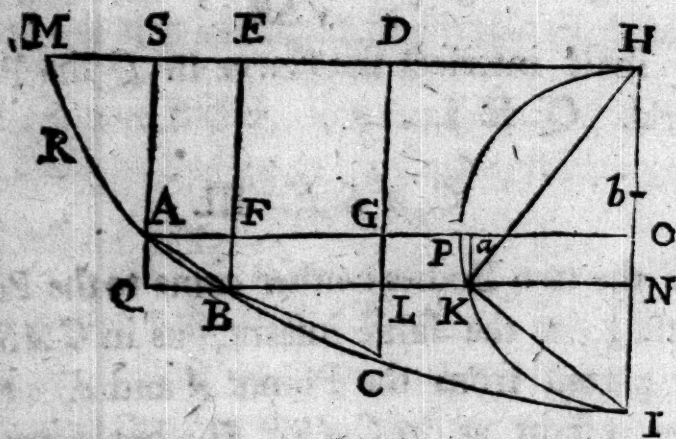
C A S E. III.

If the two Bodies neither came to the Points B and g , at the same instant, as in *CASE I*, nor parted from the Points A and d , at the same instant as in *CASE II*. but from any other Points given, in the Lines AB , dT ; by the former ways of reasoning may compute in what Points of their respective Curves they shall be found in the *same instant*: And consequently we may determine all the Distances and Positions of Bodies describing any of these Paraboloids, with respect to one another.

P R O B. XIV. F I G. XXIV.

Let it be required to find the Curve of swiftest Descent; or the nature of a Curve, in which a heavy Body descending by the force of its own Gravity, shall move from one given Point to another Point (given also) in the shortest Time, or in less Time than
in

in any other Line passing between those Points.



Suppose AB and BC to be two Infinitely little Subtenses of the Curve Line required ; which Curve is to be conceived to begin at the Point *M* (*ex.gr.*) somewhere in the Horizontal Line MH. Draw EB, DC, perpendicular to MH, and AG, QL parallel to MH (and consequently at Right Angles to EB, DC) and let the Point B be such, that the Parallel QBL carried through it, may bisect GC in the Point, L that is, may make the Lineolæ GL, CL, equal. Now the Curve ABC is suppos'd to be of such a nature, that a heavy Body moving therein, and arrived in its Descent (from the Horizontal Line MH) at any Point, as *A*, shall pass from that Point *A* to any other Point, as *C*, in the *least or shortest Time possible*. So that the Points *A* and *C* being given or determin'd, the Point *B* is to be found) and so the Nature and Property

perty of that Curve, to which this condition agrees; the two Subtenses AB and BC are in such a manner inclin'd to one another, that the Time of descent through them is the least possible.

'Tis true, one part of what is here suppos'd, may be suppos'd of any other kind of Curve, viz. that the Point *B* may be such that the Parallel BL carried through it, may bisect GC. But then, that together with this Supposition, the Point *B* is such, as that the Subtenses BA, BC (drawn from thence to two given Points as *A* and *C*) are in such a manner inclined to each other, that the time of descent through those Subtenses is of all the *least possible*; this is the result of a peculiar sort of Curvature, and into the nature of the Curve which is thus bent, we are now to enquire farther.

The Points *A* and *C* being determin'd, the Lineolæ QL or AG, will be *constant*. Also the Lineola AQ = BF = GL = CL, is *constant*; and so is the Line CD, intercepted between the given Point *C* and the Horizontal Line MH. Again, because the Point *B* is suppos'd to be such, that the parallel QL carried thro' it, bisects GC in *L*, and there is but one Point alone, that can answer that demand; therefore the Line EB intercepted between the Horizontal Line and that *only Point*, must be represented by a *constant* Quantity. Lastly, because

Cause the Lines FB, LC, being equal, the Lines AF, BL, will be unequal, and consequently AB, BC unequal too; therefore AF, BL, AB, BC, must be represented by *indeterminate* or *variable* Quantities. These necessary things premis'd, let $EB=a$. $DC=b$. $LF=LC=e$. $AF=f$. BL

$=b$. Therefore $AG=f+b$. $AB=\sqrt{ff+ee}$.

$BC=\sqrt{bb+ee}$. When the heavy Body is in B, its Velocity is the same as that acquir'd by the perpendicular Descent thro' EB; and so when 'tis in C, 'tis the same as that acquir'd by the Descent through DC. But these Velocities (by the known Law) are in the Subduplicate Ratio of those perpendicular Descents, that is as $\sqrt{EB}$: $\sqrt{DC}$, or as $\sqrt{a}$: $\sqrt{b}$. Also the Lineolæ AB, BC, being infinitely small, the Velocity throughout the whole Lineola AB may be taken for $\sqrt{a}$, and that throughout the whole Lineola CD may be taken for $\sqrt{b}$. Therefore the Times of the heavy Bodies Descent thro' the Lineolæ AB and BC (which Times are as the Spaces directly, and the Velocities reciprocally) will be express'd by $\frac{\sqrt{ff+ee}}{\sqrt{a}}$ and

$\frac{\sqrt{bb+ee}}{\sqrt{b}}$. Now the Sum of these Times (by

the Hypothesis) is a *Minimum*, that is $\frac{\sqrt{ff+ee}}{\sqrt{a}}$

+

$+\frac{\sqrt{bb+ee}}{\sqrt{b}}$ is a *Minimum*, and consequently

the Fluxion of the Expression equal to nothing; that is, (according to the *direct Method*)

$$\frac{f\dot{f}}{\sqrt{ff+ee}\sqrt{a}} + \frac{b\dot{b}}{\sqrt{bb+ee}\sqrt{b}} = 0, \text{ from}$$

$$\text{whence, } \frac{f\dot{f}}{\sqrt{ff+ee}\sqrt{a}} = \frac{-b\dot{b}}{\sqrt{bb+ee}\sqrt{b}}. \text{ But}$$

AG or $f+b$ is *constant*; therefore $\dot{f}+\dot{b}=0$, and so $\dot{f}=-\dot{b}$, and consequently dividing both sides by $+\dot{f}$ or $-\dot{b}$, and reducing, we come to this proportion, *viz*:

$\sqrt{bb+ee} : \sqrt{ff+ee} :: b \times \sqrt{a} : f \times \sqrt{b}$, that is,
 BC: AB:: BL $\times$ $\sqrt{EB}$: AF $\times$ $\sqrt{DC}$. So that making ME, and MD (in the Horizontal Line) *Abscisses*, and EB, CD their correspondent *Ordinates*; (in which case also, AF, BL are the *Incrementa* of the *Abscisses*, BF and CL those of the *Ordinates*, and AB, BC, those of the *Curviline Segments*) it follows, that the property of the Curve of swiftest Descent is this, that the Fluxions of the *Curve*, are in the Ratio compounded of the *direct* of the Fluxions of the *Abscisses*, and the *reciprocal* Subduplicate Ratio of the *Ordinates*; and therefore that Curve to which this Property agrees, must be the Curve of *swiftest Descent*. Now that this Property agrees to the *common Cycloid*, is thus concisely made out.

Imagine

Imagine the Curve MABCI (without any regard to the matter of swiftest Descent) to be the vulgar Cycloid, whose Basis is the Line MH, and consequently its Vertex I ; then supposing all as before, I say, that this Curve affords us the following

T H E O R E M.

The Fluxion of the Curve Line are universally in the Ratio compounded, of the direct Ratio of the Fluxions of the Abscisses, and the reciprocal subduplicate Ratio of the Ordinates ; the Abscisses being taken in the Base of the Cycloid, and the Ordinates at right Angles thereto.

Let HPI be the generating Semicircle, whose Diameter is HI, which is also the Axis of the Semicicloid ; and let AG, BL, be produc'd to the Points O and N.

Put $HI = d$. $MH = a$. Semicircle $HPI = p$. Arch $KI = s$. Curve $BCI = v$. $BN = z$. $KN = y$. $IN = x$. Then $EB (= HN) = d - x$, and $ME (= MH - BN) = p - z$, by the Symbols.

From the Generation of the Cycloid, $a = p$; and $z = y + s$; that is the Base of the Cycloid is = the $\frac{1}{2}$ Periphery, and any Ordinate = the

the Sum of the correspondent Circular Ordinate and Arch. Therefore $\dot{z} = \dot{y} + \dot{x}$. But

from the Nature of the Circle $y = \frac{\frac{1}{2} d\dot{x} - x\ddot{x}}{y}$

and $\dot{y} = \frac{\frac{1}{2} d\dot{x}}{y}$; and so $z\dot{z} = \frac{x^2\ddot{x} - 2dx\ddot{x} + dd\dot{x}\dot{x}}{dx - xx}$

and $\dot{z}\dot{z} + x\ddot{x} (=v\dot{v}) = \frac{x^2\ddot{x} - 2dx\ddot{x} + dd\dot{x}\dot{x}}{dx - xx}$

+ $\dot{x}\dot{x} = \frac{dd\dot{x}\dot{x} - d\dot{x}\ddot{x}}{dx - xx} = \frac{d\dot{x}\dot{x}}{x}$; and there-

fore $\dot{v}$ (the Fluxion of the Curve it self) =

$\frac{d^{\frac{1}{2}}\dot{x}}{x^{\frac{1}{2}}} = d^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$. Thus then we have found

the Expression of the Fluxion of the Curve. Now by the Theorem, this Expression is to be in the Ratio of or as, another Expression, whose Numerator must be the Fluxion of the Abscisse EM and its Denominator the Square Root of the Ordinate EB. And this is evidently so. For the Fluxion of the Abscisse ME = $-\dot{x} = -\dot{y} - \dot{x}$

$\dot{y}$, is $\frac{x\dot{x} - d\dot{x}}{y} = \frac{x\dot{x} - d\dot{x}}{\sqrt{dx - xx}}$; and the Square

Root of the Ordinate, is $\sqrt{d - x}$. Now the former of these divided by the latter, comes to

$\frac{x\dot{x} - d\dot{x}}{\sqrt{ddx - 2dx\dot{x} + xxx}} = \frac{x\dot{x} - d\dot{x}}{\sqrt{xxx - d}} = \frac{\dot{x}}{\sqrt{x}} = x^{-\frac{1}{2}}\dot{x}$.

And this Expression is to the Expression of the Fluxion of the Curve (viz. $d^{\frac{1}{2}}x^{-\frac{1}{2}}\dot{x}$) in the constant

constant Ratio of 1 to $d^{\frac{1}{2}}$. Therefore the Theorem is certain. And from hence, lastly, we may conclude, that *the Curve of swiftest Descent is a Cycloid*. Q. E. I.

S C H O L.

If a Ray of Light fell upon a Medium so constituted, that the *Rarities* of the Medium, taken at any depths, were continually in the Subduplicate Ratio of those Depths; the Ray in its Passage through such a Medium, would be bent into the Curve of *Cycloid*; and consequently the *Curve of Refraction* thus form'd, is the same with the *Curve of swiftest Descent*. The reason of which, in a few words, is this. That the Ray or Particle of Light in the one Case, observes in its motion (according to this Hypothesis) exactly the same Law; that a heavy Body descending by its Gravity, does in the other; and therefore the *Track of the Motion*, or the Curve describ'd, must necessarily be the same in both cases. Now that the very same Law is observed by the Particle of Light, and the heavy Body. is thus made out. In the *Descent of the heavy Body*, the Velocities are in the Subduplicate Ratio of the perpendicular Descents or Depths below the Horizontal Line. And in the Motion of the Particle of Light, the *Rarities* of the Medium are (by the Hypothesis)

thesis) in the Subduplicate Ratio of the Depths or perpendicular Descents. But the Velocities of the Light, are ever proportional to the Rarities of the Mediums it passes thro'; and consequently the Velocities of the Light will be in the Subduplicate Ratio of the perpendicular Descents. So that since the Velocity of the Light is continually augmented by the *encreasing Rarity of the Medium*, in the same proportions, that the Velocity of the heavy Body is augmented in, *by its Gravity*, 'tis plain that the Curve describ'd is either way the same.

All this (with some other things relating to this matter) might be shewn by a direct Calculus, but I must forbear that now, and content myself with a verbal Demonstration. I'll only note with respect to the foregoing Process.

1. The *Rectification of the Cycloid Line* is very obvious.

For the Fluxion of that Curve, we have there found, to be $\dot{v} = d^{\frac{1}{2}}x - \frac{1}{2}x$; and therefore the flowing Quantity will = the length of the Curve Line it self. That is, $2d^{\frac{1}{2}}x^{\frac{1}{2}} = v = \text{Curve Line BCI}$. But from the Nature of the Circle, the Subtense IK is $= d^{\frac{1}{2}}x^{\frac{1}{2}}$ or $\sqrt{dx}$. Therefore the Segments Curve of the Cycloid, are double the Correspondent Subtenses in the Circle. And consequently the Semicicloid it
Q
self

self, is double the Diameter, which is the Subtense of 180 gr.

2. And from hence it follows, that the Segments of the Cycloidick Linetaken towards the Base (as *ex. gr.* the Segment MRB) are equal to the double differences between the Diameter and the Circular Subtenses, belonging to the other Segments towards the Vertex.

Thus is Curve MRB = 2HI — 2IK. For Curve MBI = 2HI, and Curve BCI = 2IK; therefore, &c.

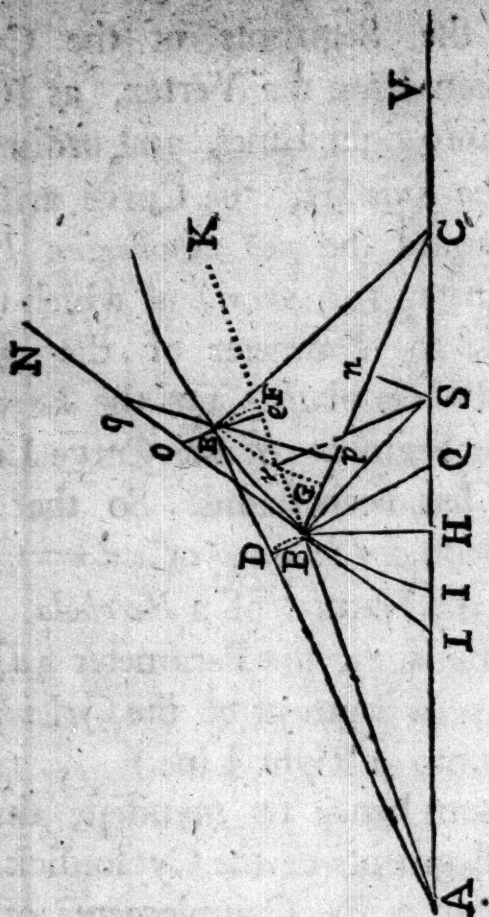
Hence 'tis no difficult Matter to cut the Curve of the Semicloid, in any given Ratio. *Ex. gr.* if the Curve were to be divided in *B*; so that the Segments ICB, BRM should be to each other in the Ratio of *m* to *n*; let the Diameter be divided in the Point *b*; so that IB : bH :: *m* : *n*. then set off IK = Ib, and thro' K draw the Ordinate NKB cutting the Cycloid in *B*: I say, that *B* is the Point that divides the Curve in the given Ratio. Algebraically, the Abscisse IN or *x*, may be thus found: Curve ICB = $2\sqrt{dx}$. Curve IBM = $2d$. Therefore Curve BRM = $2d - 2\sqrt{dx}$. Now 'tis requir'd that $2\sqrt{dx} : 2d - 2\sqrt{dx} :: m : n$, therefore $2md = 2n\sqrt{dx} + 2m\sqrt{dx}$, or $\sqrt{dx} = \frac{md}{n+m}$, and $dx = \frac{mmd}{nn+2nm+mm}$, whence $x = \frac{mmd}{n^2+2nm+m^2}$. From whence the foregoing Construction is easily deduc'd. 3. If

3. If the Segments of the Cycloidick Line (taken from the Vertex, as ICB) were extended into right Lines, and ordinately applied to the Axis IH, the Curve arising from thence would be an *Apollonian Parabola*, the Parameter (*ad Axem*) of which would be Quadruple the Diameter of the generating Circle. This is plain; for the Expression of any such Segment of the Curve Line, is $= 2\sqrt{d}x$ as has been found. So that we have this Equation $2\sqrt{d}x=v$, or $4dx=vv$, which expresses the Nature of a *Parabola*, in which the Abscisse is x , the Parameter $4d$, and the Ordinate v (a Segment of the *Cycloidick* Curve extended into a Right Line.)

And from hence 'tis manifest, that the remaining Segments of the Cycloidick Line, as BRM, that is the Complements of the Segments ICB, to the whole Curve IBM; that these, I say, will in such a *Parabola* be the Complements of the *Parabolick Ordinates*, to that Base; or (in Dr. *Wallis's* Stile) a Series of Equals less a Series of Subsecundans.

PROB. XV. FIG. XXV.

To find the Foci of Optick Glasses, universally.



The Method I shall propose here, is very pertinent to the Subject we are treating of, as being the pure Result of an Operation in Fluxions. And I believe one Example or two of it will be sufficient to shew, that it may with ease be accommodated to Glasses of all Figures receiving Rays in any kind whatsoever.

LEMMA

L E M M A.

The ultimate Ratio, of the Increment of the Incident Ray, to the Decrement of the Refracted Ray, is always accurately in the proportion of the Sine of the Angle of Incidence, to the Sine of the Refracted Angle.

Suppose the Ray AB , falling from the radiating Point A upon the refracting Curve Surface IBE , at the Point B , where 'tis refracted into BC , meeting with the Axis AI in the Point C . Let AE be another Ray, falling upon the Glass at the Point E , which is suppos'd infinitely near to B ; which is refracted into EC . Let LBN touch the Curve at the Point B , and BS be drawn at Right Angles thereto, and consequently to the Curve at that Point B . Lastly, on the Centers A and C with the Radii AE , CE , describe the little Arches EF , EG , cutting the incident and refracted Rays AB and BC , in the Points F and G .

If on the Center A with the Radius AB we describe the little Arch BD cutting the Ray AE in D ; 'tis plain that DE or BF is the *Increment* of the Incident Ray AB . Also BG is the *Decrement* of the refracted Ray BC . Let

the Sine of the Angle of Incidence, be to the Sine of the refracted Angle, as $m:n$; then, I say, that the ultimate Ratio of BF to BG, is the same with the Ratio of m to n .

Draw Ee , Ep the right Sines of the little Arches EF, EG, respectively; which produce till they intersect the Tangent LBN, in the Points o and q ; and from the Point S , let fall the perpendiculars Sr , and Sn , upon the incident and refracted Rays, BFK and BGC.

We are to shew now, that the ultimate Ratio of BF, to BG, will be that of m to n .

When the Points B and E in the Curve IBE come together, then the Points e and F , in the Lineola BF, and the Points G and p , in the Lineola Bp will also coincide; which is too plain to need any Proof. Therefore since the Lineolæ Be , Bp , are at last coincident with the Lineolæ BF, BG, if we demonstrate the ultimate Ratio of the Lineolæ Be , Bp , to be that of m to n , we then demonstrate the ultimate Ratio of BF to BG, to be that of m to n . In the Triangles Beo , BrS , Rectangular at e and r ; the Angle $Boe = rBS$ (which is the Angle of Incidence) for $Boe + eBo =$ a right one $= eBo + rBS = SBo$. Therefore from hence the Triangles Beo , BrS are similar. In like manner, in the Triangles Bpq , BnS , Rectangular at p and n ; the Angle $Bqp = nBS$ (which is the Refracted Angle) for $Bqp + pBq =$ a right

right one = $pBq + nBS = SB\sigma$. Therefore from hence the Triangle Bpq , BnS , are similar.

Hence then it follows, that $Be : B\sigma :: rS : BS$, as also $Bp : Bq :: nS : BS$. Therefore $Be :: Bp :: \frac{B\sigma \times rS}{BS} : \frac{Bq \times nS}{BS} :: B\sigma \times rS : Bq \times nS$

But when the Points B , and E , do come together, then the Points e and p , σ and q , do all coincide with and in the Point B ; therefore then the Lineolæ $B\sigma$ and Bq will be equal. Therefore the ultimate Ratio of Be to Bp , shall be that of rS to nS ; but rS and nS are the Sines of the Angle of Incidence and the refracted Angle, to the same Radius BS ; consequently $rS : nS :: m : n$; and therefore the ultimate Ratio of Be to Bp , that is, of BF to BG , shall be that of m to n . So that the ultimate Ratio of the *Increment* of the incident Ray, to the *Decrement*, &c. Q. E. D.

C O R O L L.

Since the Lineolæ BF and BG are generated in the same particle of Time; therefore the ultimate Ratio of those Lineolæ, is the Ratio of the Fluxions of the Incident and refracted Rays, AB and BC ; as appears by what was demonstrated at SECT. I. And therefore, 'tis manifest, that the *Fluxion* of the Incident Ray, is to the *Fluxion* of the Refracted Ray, as the Sine of the Angle of Incidence, to the Sine of

the *Refracted* Angle that is, as m to n . But farther, since the *Refracted* Ray BC decreases, while the *Incident* Ray AB increases; and since it has been shewn, that the Fluxion of a flowing Quantity that decreases, must have a Negative Sign, while the Fluxion of another flowing Quantity that increases, has a positive Sign. Therefore it follows, that the Fluxion of the *Incident* Ray with a positive Sign, is to the Fluxion of the *Refracted* Ray with a Negative Sign, as m to n .

These things premis'd, we come now to determine the distance of the *Focus* C , from the Vertex of the Glass I ; the radiating Point being suppos'd to be A .

Put $AI=d$. $IC=f$. $BH=y$. $AB=z$. $BC=v$. $IH=x$. Therefore $HC=f-x$. $AH=d+x$. Also from the Rectangular Triangles AHB , CHB , we have $z=\sqrt{yy+dd+2dx+xx}$, and $v=\sqrt{yy+ff-2fx+xx}$. And by the direct Method $\dot{z}=\frac{y\dot{y}+d\dot{x}+x\dot{x}}{\sqrt{yy+dd+2dx+xx}}$, and $\dot{v}=\frac{y\dot{y}-f\dot{x}+x\dot{x}}{\sqrt{yy+ff-2fx+xx}}$. From the Coroll. to the Lemma foregoing, it appears, that $\dot{z}:-\dot{v}::m:n$, that is, $\frac{y\dot{y}+d\dot{x}+x\dot{x}}{\sqrt{yy+dd+2dx+xx}}:\frac{f\dot{x}-y\dot{y}-x\dot{x}}{\sqrt{yy+ff-2fx+xx}}::m:n$, which general

Analogy

Analogy is to be determin'd to particular Curves, according to their Nature. Suppose now the Curve IBE were a Circle, whose Diameter = $2r$. Then $y\dot{y} = r\dot{x} - x\dot{x}$, which value substituted in the Analogy instead of $y\dot{y}$ and $-r\dot{x} + x\dot{x}$ instead of $-y\dot{y}$, and $2rx - xx$ instead of yy ; we have then, $\frac{d\dot{x} + r\dot{x}}{\sqrt{2rx + dd + 2dx}}$

$\frac{f\dot{x} - nr\dot{x}}{\sqrt{2rx + ff - 2fx}} :: m : n$. Multiply the Terms, for an Equation, and divide by $\dot{x}$, and then we have $\frac{nd + nr}{\sqrt{2rx + dd + 2dx}} = \frac{mf - mr}{\sqrt{2rx + ff - 2fx}}$.

Since the Focus we enquire after, is that which is made by the Rays that fall near the Vertex of the Glass; let therefore the Quantity x vanish, and then all the Terms multiplied into it, go out of the Equation. Consequently in that case, we have $\frac{nd + rn}{\sqrt{dd}} = \frac{mf - mr}{\sqrt{ff}}$,

that is, $\frac{nd + nr}{d} = \frac{mf - mr}{f}$. Whence this Equation, $m df - n df - nrf = mdr$; therefore $f = \frac{mdr}{md - nr - nd}$ Q. E. I.

And from hence the Focus may be found for all Cases of parallel and converging Rays falling on spherick Glasses; and I need not say,

say, that the same sort of Process, by which we have here determin'd the Focus, for one single Refraction, will as easily help us to find it after two or more Refractions: All these little Particulars, I must leave to the young Geometer's own Industry to work out by himself, for my narrow Bounds will not allow me here to bring them in.

But to find the Theorems for *Elliptical* and *Hyperbolical* Glasses; let the Curve IBE be imagin'd to be an *Ellipsis*, whose Parameter = a , and its Transverse Axis = b . Then will

$$yy = \frac{bax - axx}{b}, \text{ and consequently } y; =$$

$$\frac{\frac{1}{2}ba\dot{x} - ax\dot{x}}{b}, \text{ and substituting this value in}$$

the room of y ; in the *general Analogy*, then we

$$\text{have, } \frac{\frac{1}{2}ba\dot{x} - ax\dot{x}}{b} + d\dot{x} + x\dot{x}$$

$$\sqrt{\frac{bax - axx}{b} + dd - 2dx + xx} :$$

$$f\dot{x} = \frac{\frac{1}{2}ba\dot{x} + ax\dot{x}}{b} - x\dot{x} :: m:n. \text{ And}$$

$$\sqrt{\frac{bax - axx}{b} + ff - 2fx + xx}$$

dividing all by $\dot{x}$, and making x vanish, and reducing the Expressions, we shall have

$$\frac{\frac{1}{2}a + d}{d} : \frac{f - \frac{1}{2}a}{f} :: m:n; \text{ whence } \frac{1}{2}naf +$$

$$ndf = mdf - \frac{1}{2}mad, \text{ and consequently } f = \frac{1}{2}$$

$\frac{\frac{1}{2} m d a}{m d - n d - \frac{1}{2} n a}$. If now we suppose the radiating Point *A* to be at an infinite Distance, that is the Ray *AB* to be parallel to the Axis *IS*, then will *d* be infinite, and so $f = \frac{\frac{1}{2} m d a}{m d - n d} = \frac{\frac{1}{2} m a}{m - n}$.

From whence it follows, that if the Ellipsis be such, that the Transverse Axis, is to the distance of the Foci, as the Sine of the Angle of Incidence to the Sine of the Refracted Angle, that is, as *m* : *n*; that then the Point *C* will be the Focus of the Ellipsis (and 'twill be that Focus too which is the farthest from the Vertex *I* :) For let *IV* be the Transverse Axis, and *Q* the Focus next the Vertex. Put *IU* = *b*. *IQ* = *b*. Therefore *QV* = *b* - *b*. And the distance between the two Foci, will be *b* - 2*b*. By the Hypothesis, 'tis *b* : *b* - 2*b* :: *m* : *n*, whence *b* : 2*b* :: *m* : *m* -

n. Therefore $\frac{b}{2b} = \frac{m}{m-n}$, and consequently f

$$\left(= \frac{\frac{1}{2} m a}{m - n} \right) = \frac{\frac{1}{2} a b}{2b} = \frac{a b}{4b}.$$

Therefore, 4 *bf* = *ab*, or $b f = \frac{a b}{4}$, that is, the Rectangle *IC* ×

IQ = the 4th part of the Figure (or of the Rectangle under the Transverse Axis and Parameter) consequently the Point *C* is the other Focus of the Ellipsis, as appears from the Conick Sections. For no other Point but the other Focus, can give this proportion.

The

The same Calculus will serve for the Hyperbolic Glasses too, only changing the Signs, according to the Nature of that Curve.

To accommodate the *General Analogy* above, found to the Discovery of the Foci in *Catoptricks*; we must consider, that in that case, the Angle of Incidence is equal to the Angle of Reflection, (or in other Words) that $m=n$. Suppose now then (that we may the less perplex the Figure with new Points and Lines) that the same Point C , which has hitherto been the *Focus by Refraction*, was now the *Focus by Reflection*; the Point of Incidence B also continuing still the same. Then 'tis manifest that the radiating Point A must in this case be somewhere on the other side of the Glass, as *ex. gr.* at the Point Q : So that the Angle of Incidence QBS , may = the Angle of Reflection SBC . Then putting QI (the distance of the radiating Point from the Vertex of the Glass) = d here QH will = $d - x$, but all the other Expressions will be just as before. Also since $m = n$, the *General Analogy* comes into this

General Equation,
$$\frac{yy - dx + xx}{\sqrt{yy + dd - 2dx + xx}} = \frac{fx - yj - xx}{\sqrt{yy + ff - 2fx + xx}}$$
. From hence now, if the Curve IBE be a *Circle*; instead of yj , substituting its value; by the same steps as above in the case of Refraction, we shall come to this

this Equation, $\frac{r-d}{d} = \frac{f-r}{f}$, from whence

$$IC = f = \frac{dr}{2d-r}.$$

If the Curve were a *Parabola*, whose Parameter $= p$, then $y\dot{y} = \frac{1}{2}p\dot{x}$, and consequently (making x vanish) this Equation arises,

$$\frac{\frac{1}{2}p\dot{x} - d\dot{x}}{d} = \frac{f\dot{x} - \frac{1}{2}p\dot{x}}{f}, \text{ that is, } \frac{p - 2d}{2d}$$

$$= \frac{2f-p}{2f}, \text{ whence by Reduction we find } f =$$

$$\frac{2dp}{8d-2p}. \text{ If we imagine now that } d \text{ were In-}$$

finite, or that the Rays were parallel to the Axis of the *Parabolick Speculum*; then $f =$

$$\frac{2dp}{8d} = \frac{1}{4}p; \text{ so that then they are reflected into}$$

that Point, which in the most strict and genuine Sence, is the *Focus* of the *Parabola*. If

the Curve be an *Ellipsis*, then using the same Symbols as before, and instead of $y\dot{y}$ substituting its Value $\frac{\frac{1}{2}ba\dot{x} - a\dot{x}x}{b}$, in the general E-

quation; by a like Series of Steps we shall come to this Equation, $\frac{1}{2}bad = 2bdf - \frac{1}{2}baf$;

whence $f = \frac{\frac{1}{2}ad}{2d - \frac{1}{2}a} = \frac{ad}{4d - a}$. Now if the

Radiating Point Q be suppos'd to be one of the Foci of the *Ellipsis*, then the Point C shall be the other Focus. For IU being $= b$, and $IQ = d$,

and

and IC or $f = \frac{ad}{4d-a}$; then shall $CU = b - f$ be
 $= b - \frac{ad}{4d-a} = \frac{4db - ab - ad}{4d-a}$. Now if CU be $= IQ$,
 that is, if $\frac{4db - ab - ad}{4d-a}$ be $= d$, or which is
 all one, if $4db - ad - ab$ be $= 4dd - ad$,
 or if (by Transposition and Division, and re-
 jecting contradictory Terms) $db - dd = \frac{ab}{4}$;
 then the Point C shall be the other Focus
 of the Ellipsis. But this is so. For $\mathcal{Q}$ being one
 Focus (by the Hypothesis) then from the na-
 ture of the Ellipsis the Rectangle $IQ \times QU$ is $=$
 a 4th part of the *Figure*, or of the Rectangle
 under the *Transverse Axis* and its *Parameter*.
 That is, in our Symbols, (since $IQ = d$, IU
 $= b$, and so $QU = b - d$) $db - dd = \frac{1}{4}ab$.
 Therefore the Point C is the other Focus.

This Calculus is in like manner easily ac-
 commodated to the Hyperbola, where twill
 be found also, that the *Rays* issuing out of one
 Focus, do converge to the other. But these
 are things here only mention'd by the by, and
 as instances of the easiness and universality of
 the Method, grounded upon the noble Lemma
 foregoing. Those in whose way it lies more
 directly to do so, may make farther and more
particular Applications of it.

SCHOL.

S C H O L.

The Properties relating to the Foci of the Conick Sections, are universal, and agree to Rays falling upon any Points in those Curves, as well as Rays that fall near the Axis. *Ex. gr.* a Ray passing out of one Focus of an Hyperbola, and falling on any Point of the Curve, will converge to the other Focus, or to the Focus of the other *opposite Hyperbola*, as is demonstrable from the nature of that Curve; and so of the rest of the Conick Sections. Whereas, the Calculus reaches only the determination of the Foci, for Rays that fall near the Axis; upon which account it is, that the Quantity x denoting the *Abscisse* of the Curve, is made to vanish. And most certain it is, that the Rays that fall near the Axis, are those that produce the strongest and most sensible Effects, and are consequently most to be regarded. 'Tis so, whether we respect the Intentions of *Sight*, or of *Burning*. And the Foci of these Rays, and of these only, are (as we see) easily attained by the Calculus. But if we should endeavour to find the Foci of Rays, that are at any finite distance from the Axis, in which case we are to bring in all the Terms that have the Quantity x multiplied into them; 'twill be easie to see how much more tedious and perplexed

plexed an Operation we shall have upon our hands. However, something this way, I know, may be performed; but I shall not trouble the young Geometer with that Speculation now.



F I N I S.

